**5. Subgroups and Their Properties**

**Definition(5-1):** Let be a group and , a non-empty subset of . Then is a subgroup of , if is itself a group.

**Definition(5-2):** Let be a group and , then is a subgroup of if,

1. ;
2. The identity element of is an element of , ;
3. .

**Remark(5-3):** Each group has at least two subgroups and , these subgroups are known trivial subgroups and improper, any subgroup different from these subgroups known proper subgroup.

**Example(5-4):** is a proper subgroup of .

**Example(5-5):** is a proper subgroup of .

**Example(5-6):** is a proper subgroup of , but not subgroup of .

**Example(5-7):**  is a subgroup of .

**Theorem(5-8):** Let be a group and , then is a subgroup of iff .

**Proof:**  let be a subgroup of and , then

let , to prove be a subgroup of

1. Since ;
2. Since and ;
3. Let and is a subgroup of .

**Example(5-9):** Let be a group and . Show that is a subgroup of .

**Solution:** let , to prove

is a subgroup of .

**Theorem(5-10):** If is the collection of subgroup of , then is also subgroup of .

**Proof:** 1. Since ;

2. let , to prove

Since

is a subgroup of .

**Theorem(5-11):** Let be the collection of subgroups of and let and such that there is and , then is also subgroup of .

**Proof:** 1. Since for some ;

2**.** let, then or , so

is a subgroup of .

**Theorem(5-12):** Let and are two subgroups of , then is a subgroup of iff or .

**Proof:**  let is a subgroup of ,

to prove or

suppose that and

and

or

or , but this is contradiction

or

let or

To prove is a subgroup of

If is a subgroup of

If is a subgroup of

is a subgroup of .

**Remark(5-13):** need not be a subgroup of , for example:

is a subgroup of

is a subgroup of

is not a subgroup of , since .

**Definition(5-14):** Let be a group and are two subgroups of , then the product of and is the set:

**Notes(5-15):**

1. is write ;
2. If , then . If , then ;
3. .

**Theorem(5-16):** Let be a group and are two subgroups of , then

1. and .
2. and .
3. is a subgroup of iff .
4. If is an abelian group, then is a subgroup of.

**Proof:**

1. and , and let , and , and .
2. Let, similarly, .
3. suppose is a subgroup of , to prove , this means and , let and , since is a subgroup of , let and, and to prove (**Homework)**.

let, to prove is a subgroup of

and (by 1)

Let , to prove

and

and

is a subgroup of .

1. , let

To prove

and

and

is a subgroup of .

**Example(5-17):** In , let and . Find .

**Solution:** .

**Note(5-18):** Let and are two subgroups of, then:

1. ;
2. need not be a subgroup of, give example (**Homework).**

**Example(5-18):** Is is a subgroup of ? (**Homework).**

**Example(5-19):** Is is a subgroup of ? (**Homework).**

**Definition(5-20):** The center of a group denoted by Cent() or is the set .

**Note(5-21):** , since .

**Example(5-22):** The group , , since is an abelian group.

**Example(5-23):** The group , , since

.

**Theorem(5-24):** Let be a group. Then is a subgroup of .

**Proof:** ,

let , to prove

To prove

is a subgroup of.

**Theorem(5-25):** Let be a group, then iff is an abelian group.

**Proof:**

is an abelian group.

suppose that is an abelian group, to prove

This means and

By definition of

To prove

Let is an abelian group

.