**2. Some Properties of Groups**

**Theorem(2-1):** If $(G,\*)$ a group, then the left and right cancellation laws hold in $G$, that is:

1. $a\*b=a\*c⟹b=c$
2. $b\*a=c\*a⟹b=c, ∀a,b,c\in G$.

**Proof:** 1. Suppose $a\*b=a\*c$, then $∃a^{-1}\in G $

$$\ni a^{-1}\*\left(a\*b\right)=a^{-1}\*\left(a\*c\right)$$

$$⟹\left(a^{-1}\*a\right)\*b=(a^{-1}\*a)\*c$$

$$⟹e\*b=e\*c$$

$⟹b=c$.

(2) **(Homework).**

**Theorem(2-2):** In a group $(G,\*)$, there is exactly one element $e$ in $G$ such that $e\*a=a\*e=a∀a\in G$.

**Proof:** Assume that $G$ has two identity elements $e$ and $e^{\*}$, this means for all $a\in G$, we have $a\*e=e\*a=a$ and $a\*e^{\*}=e^{\*}\*a=a$

 $e\*e^{\*}=e^{\*}\*e=e$ and $e^{\*}\*e=e\*e^{\*}=e^{\*}⟹e=e^{\*}$.

**Theorem(2-3):** In a group $(G,\*)$, the inverse element of each element of $G$ is a unique.

**Proof:** Let$a\in G$ and $a$ has two inverses $x$ and$x^{\*}$, such that

$a\*x=x\*a=e$ and $a\*x^{\*}=x^{\*}\*a=e$

$⟹x=x\*e=x\*\left(a\*x^{\*}\right)=\left(x\*a\right)\*x^{\*}=e\*x^{\*}=x^{\*}$.

**Theorem(2-4):** If $(G,\*)$ is a group, then

1. $e^{-1}=e$
2. $(a^{-1})^{-1}=a ∀a\in G$
3. $(a\*b)^{-1}=b^{-1}\*a^{-1} ∀a,b\in G$

**Proof:** 1. Let $e^{-1}=x$

$$x\*e=e\*x=x…1$$

$$e\*x=x\*e=e…2$$

From 1 and 2, $x=e⟹e^{-1}=e$.

(2) $\left(a^{-1}\right)^{-1}=\left(a^{-1}\right)^{-1}\*e=\left(a^{-1}\right)^{-1}\*\left(a^{-1}\*a\right)$

 $=\left(\left(a^{-1}\right)^{-1}\*a^{-1}\right)\*a=e\*a=a$.

(3) since $(a\*b)\in G⟹(a\*b)^{-1}\in G$

$$\left(a\*b\right)\*\left(a\*b\right)^{-1}=\left(a\*b\right)^{-1}\*\left(a\*b\right)=e$$

$$\left(a\*b\right)\*\left(a\*b\right)^{-1}=e$$

$$a^{-1}\*\left(a\*b\right)\*\left(a\*b\right)^{-1}=a^{-1}\*e$$

$$(a^{-1}\*a)\*b\*\left(a\*b\right)^{-1}=a^{-1}$$

$$e\*b\*\left(a\*b\right)^{-1}=a^{-1}$$

$$b^{-1}\*b\*\left(a\*b\right)^{-1}=b^{-1}\*a^{-1}$$

$$e\*\left(a\*b\right)^{-1}=b^{-1}\*a^{-1}$$

$\left(a\*b\right)^{-1}=b^{-1}\*a^{-1}$.

**Theorem(2-5):** Let $(G,\*)$ be a group, then

1. $\left(a\*b\right)^{-1}=a^{-1}\*b^{-1}$ iff $G$ is an abelian group.
2. If $a=a^{-1}$, then $G$ is an abelian group.

**Proof:** i. $(⟹)$ let $(G,\*)$ be a group and $\left(a\*b\right)^{-1}=a^{-1}\*b^{-1}$

To prove $(G,\*)$ is an abelian group.

Let $a,b\in G$, to prove $a\*b=b\*a ∀a,b\in G$

$$a\*b=\left(\left(a\*b\right)^{-1}\right)^{-1}$$

$$ =(b^{-1}\*a^{-1})^{-1}$$

$$ =\left(b^{-1}\right)^{-1}\*\left(a^{-1}\right)^{-1}$$

$$ =b\*a$$

$(⟸)$ let $(G,\*)$ be an abelian group, to prove $\left(a\*b\right)^{-1}=a^{-1}\*b^{-1}$

$\left(a\*b\right)^{-1}=b^{-1}\*a^{-1}=a^{-1}\*b^{-1}$.

(ii) let $a=a^{-1}$,

to prove $a\*b=b\*a ∀a,b\in G$

$a\*b=\left(a\*b\right)^{-1}=b^{-1}\*a^{-1}=b\*a$.

**Remark(2-6):** The converse of above part is not true, for example let $(G=\left\{1,-1,i,-i\right\},∙)$ be an abelian group with $a=i⟹a^{-1}=-i⟹a\ne a^{-1}$.

**Theorem(2-7):** In a group $(G,\*)$, the equations $a\*x=b$ and $y\*a=b$ have a unique solutions.

**Proof:** $a\*x=b$

$$⟹a^{-1}\*\left(a\*x\right)=a^{-1}\*b$$

$$⟹\left(a^{-1}\*a\right)\*x=a^{-1}\*b$$

$$⟹e\*x=a^{-1}\*b$$

$$⟹x=a^{-1}\*b$$

 To show the solution is a unique

 Let $x^{\*}\in G\ni a\*x^{\*}=b$

$$ ⟹a\*x^{\*}=a\*x$$

 $⟹x^{\*}=x$.

The proof of $y\*a=b$ (**Homework).**