APPROXIMATE SOLUTION OF FUZZY INITIAL VALUE PROBLEMS USING VARIATIONAL ITERATION METHOD

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Abstract

The main objective of this paper is to solve fuzzy initial value problems, in which the fuzziness occur in the initial conditions. Two approaches are used to find the approximate solution, namely the variational iteration method and the Laplace variational iteration method. From the obtained results, as it is expected, the approximate solution of the variational iteration method with Laplace transformation is more a curate than those results obtained without using Laplace transformation.

Basic concepts

The basic definition of fuzzy numbers is given in [1]
We denote the set of all real numbers by R and the set of all fuzzy numbers on R is indicated by E. A fuzzy number is a
mapping $u: \mathbb{R} \rightarrow [0,1]$ with the following properties:
(a) u is upper semi-continuous,
(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$
(c) u is normal, i.e., $\exists\, x_0 \in \mathbb{R}$ for which $u(x_0) = 1$
(d) $\text{supp } u = \{x \in \mathbb{R} | u(x) > 0\}$ is the support of the u, and its closure cl (supp u) is compact.
An equivalent parametric definition is also given in [16,22,36] as follows:
Definition 2.1. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions$(\underline{u}(r), \overline{u}(r)$,$0 \leq r \leq 1$ which satisfy the
following requirements:
1. $\underline{u}(r)$is a bounded non-decreasing left continuous function in (0, 1], and right continuous at 0,
2. $\overline{u}(r)$ is a bounded non-increasing left continuous function in (0, 1], and right continuous at 0,
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$

A crisp number x is simply represented by $(\underline{u}(r), \overline{u}(r)) = (x, x)$, $0 \le r \le 1$. By appropriate definitions, the fuzzy number space $\{(\underline{u}(r), \overline{u}(r))\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space

Definition [1] Let $x = (\underline{x}(r), \overline{x}(r))$ and $y = (\underline{y}(r), \overline{y}(r)), \in E^1, 0 \le r \le 1$ and arbitrary $k \in R$.

Then

(1) $x = y$ iff $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$

(2) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$,

(3) $x - y = ((\underline{x}(r) - \underline{y}(r), , \overline{x}(r) - \overline{y}(r))$,

(4) $kx = \begin{cases} (k\underline{x}(r), k\overline{x}(r), k \ge 0 \\ (k\overline{x}(r), k\underline{x}(r), k < 0 \end{cases}$

Definition[2] For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v}) \in E^1$, the quantity

$$D(u, v) = \left(\int_0^1 ((\underline{u}(r) - \underline{v}(r))^2 dr + \int_0^1 ((\overline{u}(r) - \overline{v}(r))^2 dr\right)^{1/2}$$

is the distance between fuzzy numbers u and v.[2].[3]

Laplace Transformation:

The Laplace transformation is an integral transform that changes a real valued function f (t) into a function F(s), which is defined as follows:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

where, in general, s is a real variable Also, after a certain problem is transformed into an algebraic equation and then solved for F(s), we need to transform F(s) back into f(t). The solution of the original problem, which is called the inverse Laplace transform of F(s) is denoted by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Among the most important properties of the Laplace transformation, that will be used later with the VIM, is the convolution of two functions f_1 and f_2. Let the functions f_1 and f_2 be defined for all $t \geq 0$, then the convolution of f_1 and f_2, denoted by $(f_1 * f_2)(t)$, and is defined by the integral

$$(f_1 * f_2)(t) = \int_0^t f_1(t - \ell) f_2(\ell) d\ell$$

Now, let $\mathcal{L}\{f_1(t)\} = F_1(s)$ and $\mathcal{L}\{f_2(t)\} = F_2(s)$, then:

$$\mathcal{L}\{\int_0^t f_1(t - \ell) f_2(\ell) d\ell\} = F_1(s) F_2(s)$$

Conversely:

$$\mathcal{L}^{-1}\{F_1(s)F_2(s)\} = \int_0^t f_1(t-\ell)f_2(\ell)d\ell$$

Also, the convolution is commutative, that is:

$$(f_1 * f_2)(t) = \int_0^t f_1(t-\ell)f_2(\ell)d\ell \quad = \int_0^t f_2(t-\ell)f_1(\ell)d\ell$$

In addition, the most two important properties of the Laplace transformation for solving differential equations is that one which transform the differential operator $\frac{df}{dt}$ into an algebraic operator $sF(s) - f(0)$

VIM for solving FODE'S

Hossein Jafari and Mohammad saeidy and Dumitru Baleanu in 2012 [4] solves n-th order fuzzy differential equations by using variational iteration method .According to our belief, the followed approach seams to be incorrect and have many difficult obstacles in solving fuzzy ordinary differential equations using the VIM .This is due to effect of the coefficients to be either positive or negative or a mixed of than, which will affect on the upper and lower solutions of FODE.

The results obtained in [4] for r=1 for the upper and lower solutions $\underline{y}$ and $\overline{y}$ are not equal which contradiction the basic theory of the solution of FODE.

In the next section , we will illustrate the most strict approach for solving such type of equations using the VIM.

VIM for solving n-th order FODE'S[5],[6],[7]

Consider the following n-th order fuzzy differential equation .[8]

$$y^{(n)}(t) + k_{n-1}y^{(n-1)}(t) + \cdots + k_1 y'(t) + k_0 y(t) = R(t) \quad , t \in [0,1]$$

With initial conditions $\ldots\ldots\ldots\ldots\ldots.(1)$

$$\tilde{y}(0) = \big(g_0(r), h_0(r)\big), \tilde{y}'(0) = \big(g_1(r), h_1(r)\big), \ldots, \tilde{y}^{(n-1)}(0) = \left(g_{(n-1)}(r), h_{(n-1)}(r)\right)$$

We have three cases

Case 1:

When all the coefficients $k_{n-1}, k_{n-2}, \ldots, k_0$ are positive.

The correction functionals for Eq(1) read:

$$\underline{y}_{n+1}(t,r) = \underline{y}_n(t,r) + \int_0^t \underline{\lambda}\left\{\underline{y_n}^{(n)}(s) + k_{n-1}\underline{y_n}^{(n-1)}(s) + \cdots + k_1\underline{y_n}'(s) + k_0\underline{y_n}(s) - \underline{R(s)}\right\} ds \;\ldots.(2)$$

$$\overline{y}_{n+1}(t,r) = \overline{y}_n(t,r) + \int_0^t \overline{\lambda}\left\{\overline{y_n}^{(n)}(s) + k_{n-1}\overline{y_n}^{(n-1)}(s) + \cdots + k_1\overline{y_n}'(s) + k_0\overline{y_n}(s) - \overline{R(t)}\right\} ds \;\ldots.(3)$$

$$n \geq 0$$

We obtain the Lagrange multipliers in the form.

$$\underline{\lambda}(s,t) = \overline{\lambda}(s,t) = (-1)^n \frac{(s-t)^{(n-1)}}{(n-1)!} \quad \ldots\ldots\ldots.(4)$$

It should be noted that if f a is a linear mapping according to the Euler-Lagrange differential equations we can obtain the exact value of the Lagrange multipliers, As a result , we get the following iteration formula.

$$\underline{y}_{n+1}(t,r) = \underline{y}_n(t,r) + \int_0^t (-1)^n \frac{(s-t)^{(n-1)}}{(n-1)!} \left\{ \underline{y_n}^{(n)}(s) + k_{n-1}\underline{y_n}^{(n-1)}(s) + \cdots + k_1 \underline{y_n}'(s) + k_0 \underline{y_n}(s) - \underline{R(s)} \right\} ds \quad \text{.....(5)}$$

$$\overline{y}_{n+1}(t,r) = \overline{y}_n(t,r) + \int_0^t (-1)^n \frac{(s-t)^{(n-1)}}{(n-1)!} \left\{ \overline{y_n}^{(n)}(s) + k_{n-1}\overline{y_n}^{(n-1)}(s) + \cdots + k_1 \overline{y_n}'(s) + k_0 \overline{y_n}(s) - \overline{R(t)} \right\} ds \quad \text{......(6)}$$

By Banach's fixed point theorem, it is easy to obtain the convergence condition for the sequences obtained from(5,6).

Case2:

When some of the coefficients

$k_{n-1}, \dots, k_{n-2}$ are positive and $k_{n-m-1}, k_{n-m-2}, \dots, k_1, k_0$ are negative.

The correction functions for Eq(1) read :

$$\underline{y}_{n+1}(t,r) = \underline{y}_n(t,r) + \int_0^t (-1)^n \frac{(s-t)^{(n-1)}}{(n-1)!} \left\{ \underline{y}_n^{(n)}(s) + k_{n-1}\underline{y}_n^{(n-1)}(s) + \cdots + k_{n-m}\underline{y}_n^{(n-m)}(s) + k_{n-m-1}\overline{y}_n^{(n-m-1)}(s) + \cdots + k_0\overline{y}_n - \underline{R(s)} \right\} ds \quad \dots\dots\dots\dots\dots..\dots\dots.(7)$$

$$\overline{y}_{n+1}(t,r) = \overline{y}_n(t,r) + \int_0^t (-1)^n \frac{(s-t)^{(n-1)}}{(n-1)!} \left\{ \overline{y_n}^{(n)}(s) + k_{n-1}\overline{y_n}^{(n-1)}(s) + \cdots + k_{n-m}\overline{y_n}^{(n-m)}(s) + k_{n-m-1}\underline{y}_n^{(n-m-1)}(s) + \cdots + k_0\underline{y_n}(s) - \overline{R(t)} \right\} ds \dots\dots\dots\dots\dots(8)$$

$$n \geq 0$$

Case3:

When all the coefficients $k_{n-1}, k_{n-2}, \dots, k_0$ are negative

The correction functionals for Eqs(1) reads:

$$\underline{y}_{n+1}(t,r) = \underline{y}_n(t,r) + \int_0^t (-1)^n \frac{(s-t)^{(n-1)}}{(n-1)!} \left\{ \underline{y}_n{}^{(n)}(s) + k_{n-1}\overline{y}_n(s)^{(n-1)}(s) + \dots + k_1 \overline{y_n}'(s) + k_0 \overline{y_n}(s) - \underline{R(s)} \right\} ds \dots\dots\dots\dots(9)$$

$$\overline{y}_{n+1}(t,r) = \overline{y}_n(t,r) + \int_0^t (-1)^n \frac{(s-t)^{(n-1)}}{(n-1)!} \left\{ \overline{y_n}^{(n)}(s) + k_{n-1}\underline{y}_n(s)^{(n-1)}(s) + \dots + k_1 \underline{y_n}'(s) + k_0 \underline{y_n}(s) - \overline{R(t)} \right\} ds \dots\dots\dots\dots(10)$$

$$n \geq 0$$

Laplace VIM for n-th order FDE'S.[9]

Consider the following Eqs(1)

Back to (VIM for n-th order FDE'S).

Also, we have three cases.

Case1:

By using the Laplace transform of the differential operator of the differential operator of the correction functional eqs (1),(2)

On may get:

$$\underline{\lambda} = \underline{\lambda}(s - t)$$

$$\bar{\lambda} = \bar{\lambda}(s - t)$$

$$\mathcal{L}\{\underline{y}_{n+1}(t,r)\} =$$

$$\mathcal{L}\left\{\underline{y}_n(t,r)\right\} + \mathcal{L}\left\{\int_0^t \underline{\lambda}\left\{\underline{y_n}^{(n)}(s) + k_{n-1}\underline{y_n}^{(n-1)}(s) + \cdots + k_1\underline{y_n}'(s) + k_0\underline{y_n}(s) - \underline{R(s)}\right\} ds\right\} \ldots .(12)$$

$$\mathcal{L}\{\overline{y}_{n+1}(t,r)\}$$

$$= \mathcal{L}\{\overline{y}_n(t,r)\} + \mathcal{L}\left\{\int_0^t \overline{\lambda}\left\{\overline{y_n}^{(n)}(s) + k_{n-1}\overline{y_n}^{(n-1)}(s) + \cdots + k_1\overline{y_n}'(s) + k_0\overline{y_n}(s) - \overline{R(t)}\right\} ds\right\} \ldots .(13)$$

There fore , upon using the convolution theorem with respect to t , eq's (12),(13) will be reduced

$$\mathcal{L}\left\{\underline{y}_{n+1}(t,r)\right\} = \mathcal{L}\left\{\underline{y}_n(t,r)\right\} + \mathcal{L}\left\{\underline{\lambda}(t) * [\underline{y_n}^{(n)}(s) + k_{n-1}\underline{y_n}^{(n-1)}(s) + \cdots + k_1\underline{y_n}'(s) + k_0\underline{y_n}(s) - \underline{R(s)]}\right\} \ldots.(14)$$

$$\mathcal{L}\{\overline{y}_{n+1}(t,r)\} = \mathcal{L}\{\overline{y}_n(t,r)\} + \mathcal{L}\{\overline{\lambda(t)} * \left\{\overline{y_n}^{(n)}(s) + k_{n-1}\overline{y_n}^{(n-1)}(s) + \cdots + k_1\overline{y_n}'(s) + k_0\overline{y_n}(s) - \overline{R(t)}\right\} \ldots.(15)$$

Which implicit that

$$\mathcal{L}\left\{\underline{y}_{n+1}(t,r)\right\} = \mathcal{L}\left\{\underline{y}_n(t,r)\right\} + \mathcal{L}\left\{\underline{\lambda}(t)\right\} \mathcal{L}\{\underline{y_n}^{(n)}(t) + k_{n-1}\underline{y_n}^{(n-1)}(t) + \cdots + k_1\underline{y_n}'(t) + k_0\underline{y_n}(t) - \underline{R(t)}\right\} \ldots.. (16)$$

$$\mathcal{L}\{\overline{y}_{n+1}(t,r)\} = \mathcal{L}\{\overline{y}_n(t,r)\} + \mathcal{L}\{\overline{\lambda(t)}\}\mathcal{L}\left\{\overline{y_n}^{(n)}(t) + k_{n-1}\overline{y_n}^{(n-1)}(t) + \cdots + k_1\overline{y_n}'(t) + k_0\overline{y_n}(t) - \overline{R(t)}\right\} \ldots.(17)$$

Taking the Laplace inverse of eq's (16)&(17)

We get , the following iteration formulations.

$$\underline{y}_{n+1}(t,r) = \underline{y}_n(t,r) + \mathcal{L}^{-1}\{\mathcal{L}\{\underline{\lambda}(t)\}\mathcal{L}\{\underline{y_n}^{(n)}(t) + k_{n-1}\underline{y_n}^{(n-1)}(t) + \cdots + k_1\underline{y_n}'(t) + k_0\underline{y_n}(t) - \underline{R(t)}\} \quad .... \quad .(18)$$

$$\overline{y}_{n+1}(t,r) = \overline{y}_n(t,r) + \mathcal{L}^{-1}\{\mathcal{L}\{\overline{\lambda}(t)\}\mathcal{L}\{\overline{y_n}^{(n)}(t) + k_{n-1}\overline{y_n}^{(n-1)}(t) + \cdots + k_1\overline{y_n}'(t) + k_0\overline{y_n}(t) - \overline{R(t)}\} \text{........}(19)$$

n=0,1,2,...

Case 2:

By using the Laplace transform eq's(7)&(8)

The same way in the first case

We get the final following iteration formulations

$$\underline{y}_{n+1}(t,r) = \underline{y}_n(t,r) + \mathcal{L}^{-1}\left\{\mathcal{L}\{\underline{\lambda}(t)\}\mathcal{L}\left\{\underline{y}_n{}^{(n)}(t) + k_{n-1}\underline{y}_n{}^{(n-1)}(t) + \cdots + k_{n-m}\underline{y}_n^{(n-m)}(t) + k_{n-m-1}\overline{y}_n^{(n-m-1)}(t) + \cdots + k_0\overline{y}_n - \underline{R(t)}\right\}\right\} \ldots\ldots(20)$$

$$\overline{y}_{n+1}(t,r) = \overline{y}_n(t,r) + \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\overline{\lambda}(t)\mathcal{L}\left\{\overline{y_n}^{(n)}(t) + k_{n-1}\overline{y_n}^{(n-1)}(t) + \cdots + k_{n-m}\overline{y_n}^{(n-m)}(t) + k_{n-m-1}\underline{y}_n^{(n-m-1)}(t) + \cdots + k_0\underline{y_n}(t) - \overline{R(t)}\right\}\right\}\right\} \ldots\ldots(21)$$

n=0,1,2, …

Case 3:

By using the Laplace transform eq's(9)&(10)

The same way in the first case

We get the final following iteration formulations.

$$\underline{y}_{n+1}(t,r) = \underline{y}_n(t,r) + \mathcal{L}^{-1}\{\mathcal{L}\{\underline{\lambda}(t)\}\mathcal{L}\{\underline{y_n}^{(n)}(t) + k_{n-1}\overline{y}_n(t)^{(n-1)} + \cdots + k_1\overline{y_n}'(t) + k_0\overline{y_n} - \underline{R(t)}\}\} \ldots \ldots \ldots \ldots (22)$$

$$\overline{y}_{n+1}(t,r) = \overline{y}_n(t,r) + \mathcal{L}^{-1}\left\{\mathcal{L}\{\overline{\lambda}(t)\}\mathcal{L}\left\{\overline{y_n}^{(n)}(t) + k_{n-1}\underline{y}_n(t)^{(n-1)} + \cdots + k_1\underline{y}_n{}'(t) + k_0\underline{y}_n(t) - \overline{R(t)}\right\}\right\} \ldots \ldots \ldots (23)$$

n=0,1,2,...

Illustrative Examples

Example 1:

Consider the following second-order fuzzy linear differential equation:

$$y'' - 4y' + 4y = 0 \ldots\ldots\ldots\ldots\ldots..(24)$$

$$y(0) = (2 + r;\ 4 - r);$$

$$y'(0) = (5 + r;\ 7 - r)$$

The exact solution is as follows:

$$\underline{y(t,r)} = (2 + r)e2t + (1 - r)te2t$$

$$\overline{y(t,r)} = (4 - r)e2t + (r - 1)te2t$$

This example is as per the second case

To apply the VIM, first we rewrite Eq. (24) in the form

$$\underline{y}_{n+1} = \underline{y}_n + \int_0^t \underline{\lambda}\left[\underline{y}_n''(s) - \overline{y}_n'(s) + \underline{y}_n(s)\right] ds.........(25)$$

$$\overline{y}_{n+1} = \overline{y}_n + \int_0^t \overline{\lambda}\left[\overline{y}_n''(s) - \underline{y}_n'(s) + \overline{y}_n(s)\right] ds.........(26)$$

$$\underline{\lambda} = \overline{\lambda} = (s - t)............(27)$$

Sub (27) in (25)&(26),we get

$$\underline{y}_{n+1} = \underline{y}_n + \int_0^t (s - t)\left[\underline{y}_n''(s) - \overline{y}_n'(s) + \underline{y}_n(s)\right] ds$$

$$\overline{y}_{n+1} = \overline{y}_n + \int_0^t (s - t)\left[\overline{y}_n''(s) - \underline{y}_n'(s) + \overline{y}_n(s)\right] ds$$

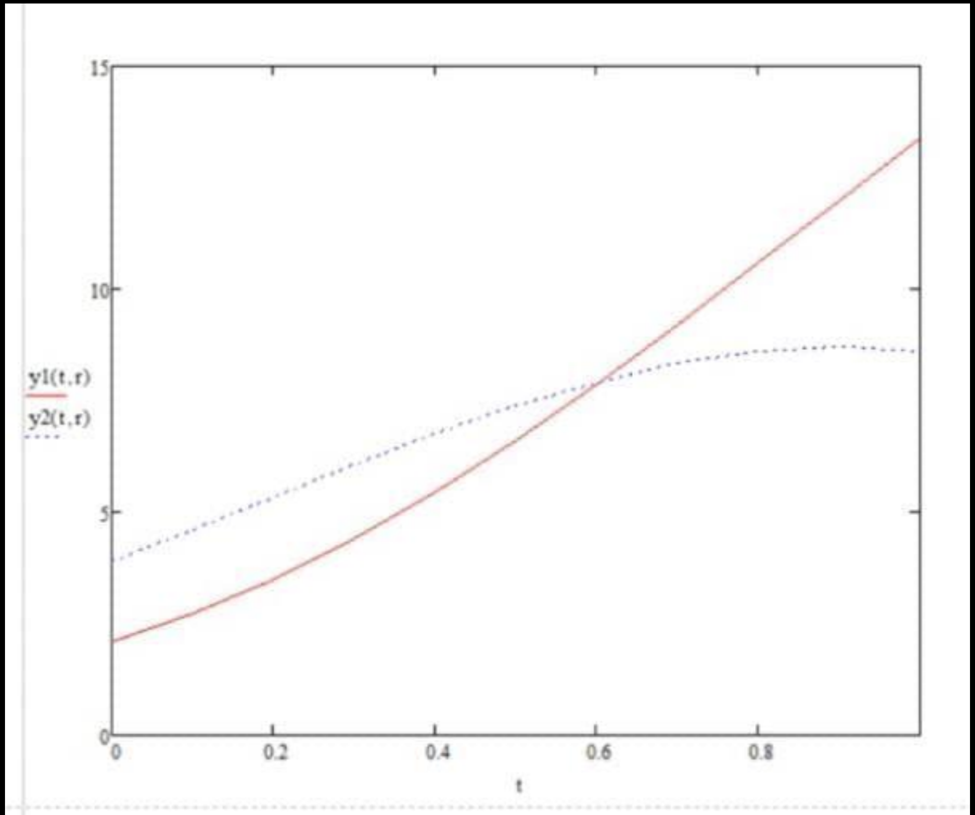
Choosing $y_0(t; r)$ = (2 + r) + t(5 + r) and $\overline{y_0}(t; r)$ = (4 − r) + t(7 − r), after 3 iterations we obtained approximate solutions

$$\underline{y} \approx 2 + r + t(5 + r) - 4rt^2 + 10t^2 - \frac{2}{3}t^3(5 + r)$$

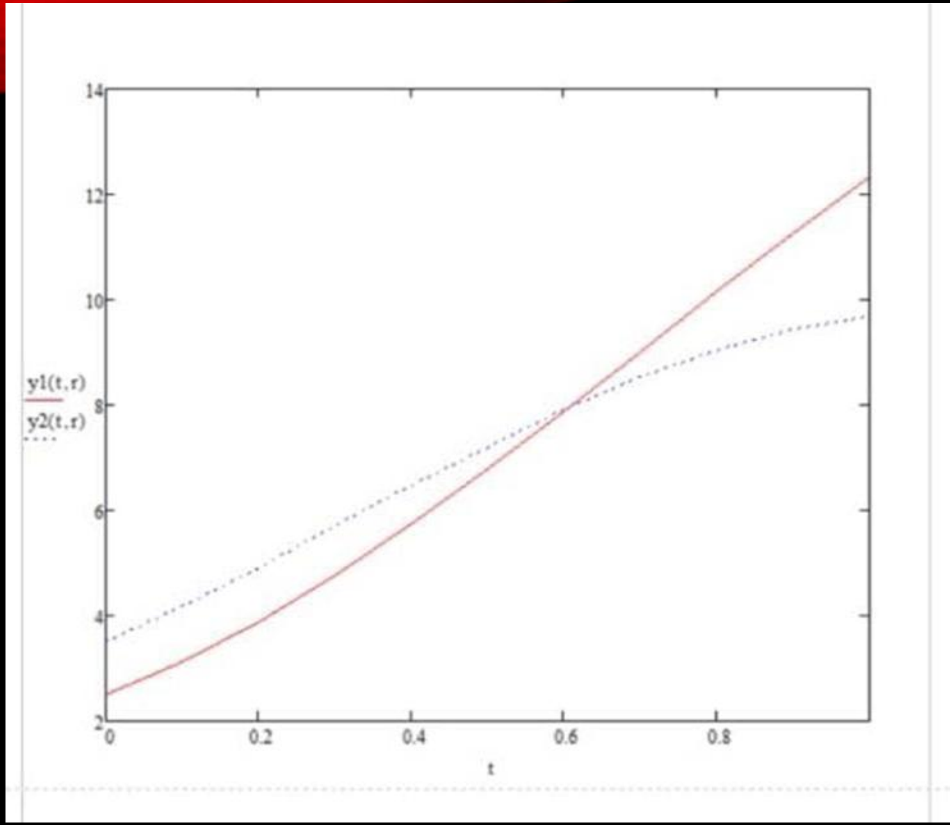
$$\bar{y} \approx 4 - r + t(7 - r) + 2t^2 + 4rt^2 - \frac{2}{3}t^3(7 - r)$$

T=0,0.1,…,1

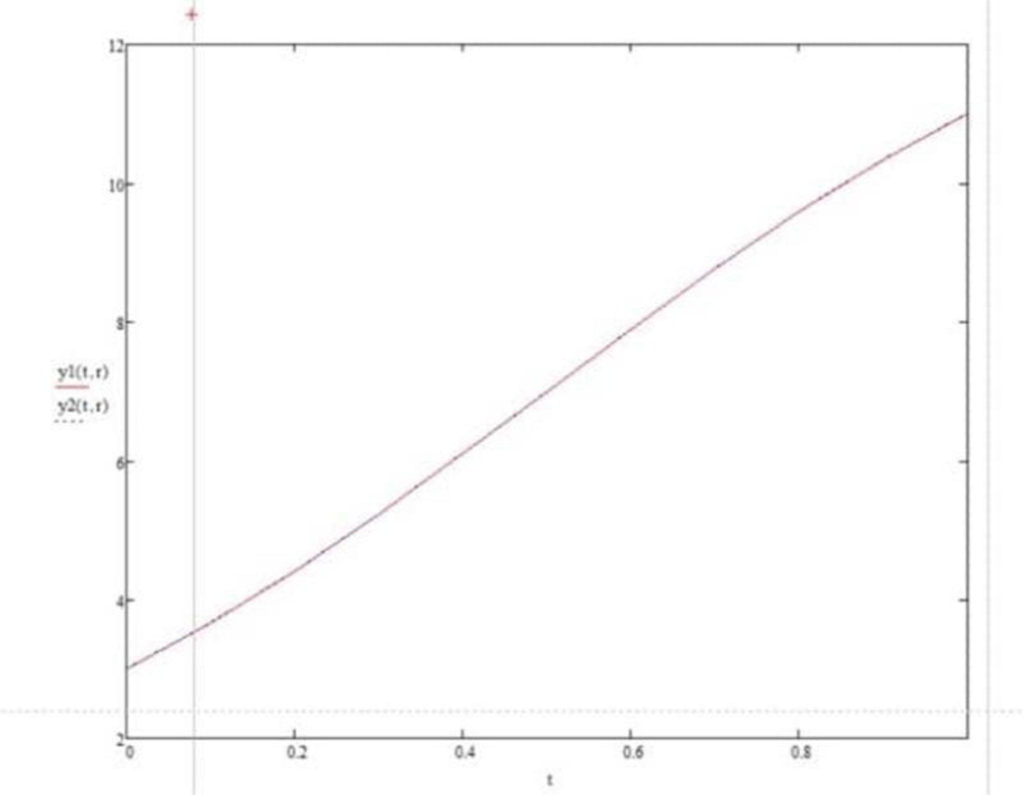
When r=0.1



When r=0.5

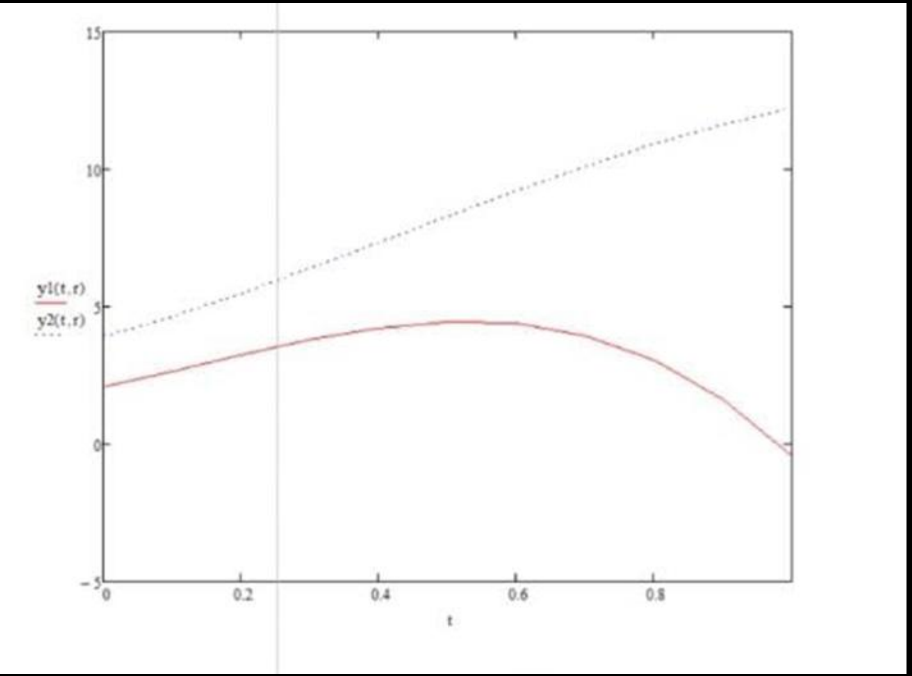


When r=1

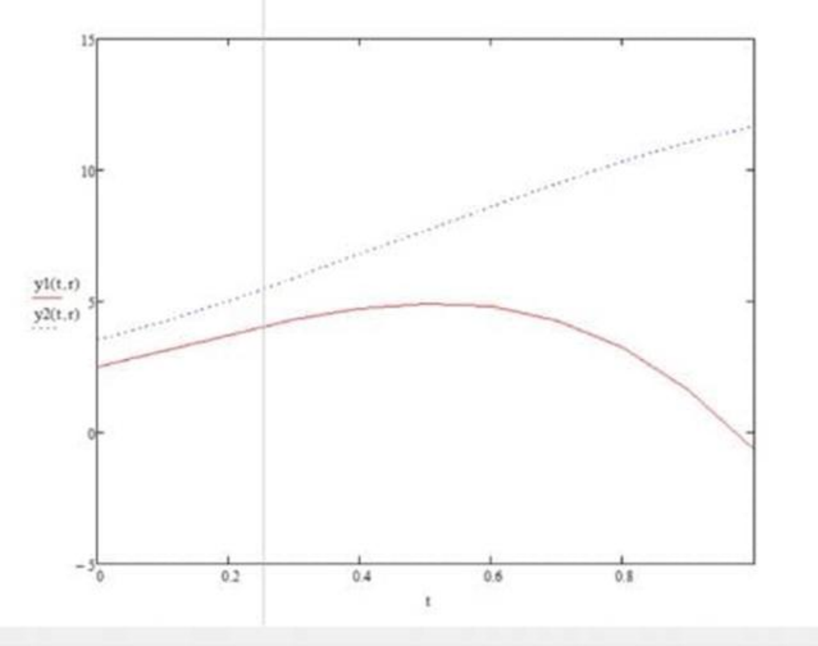


While the results are in [4]

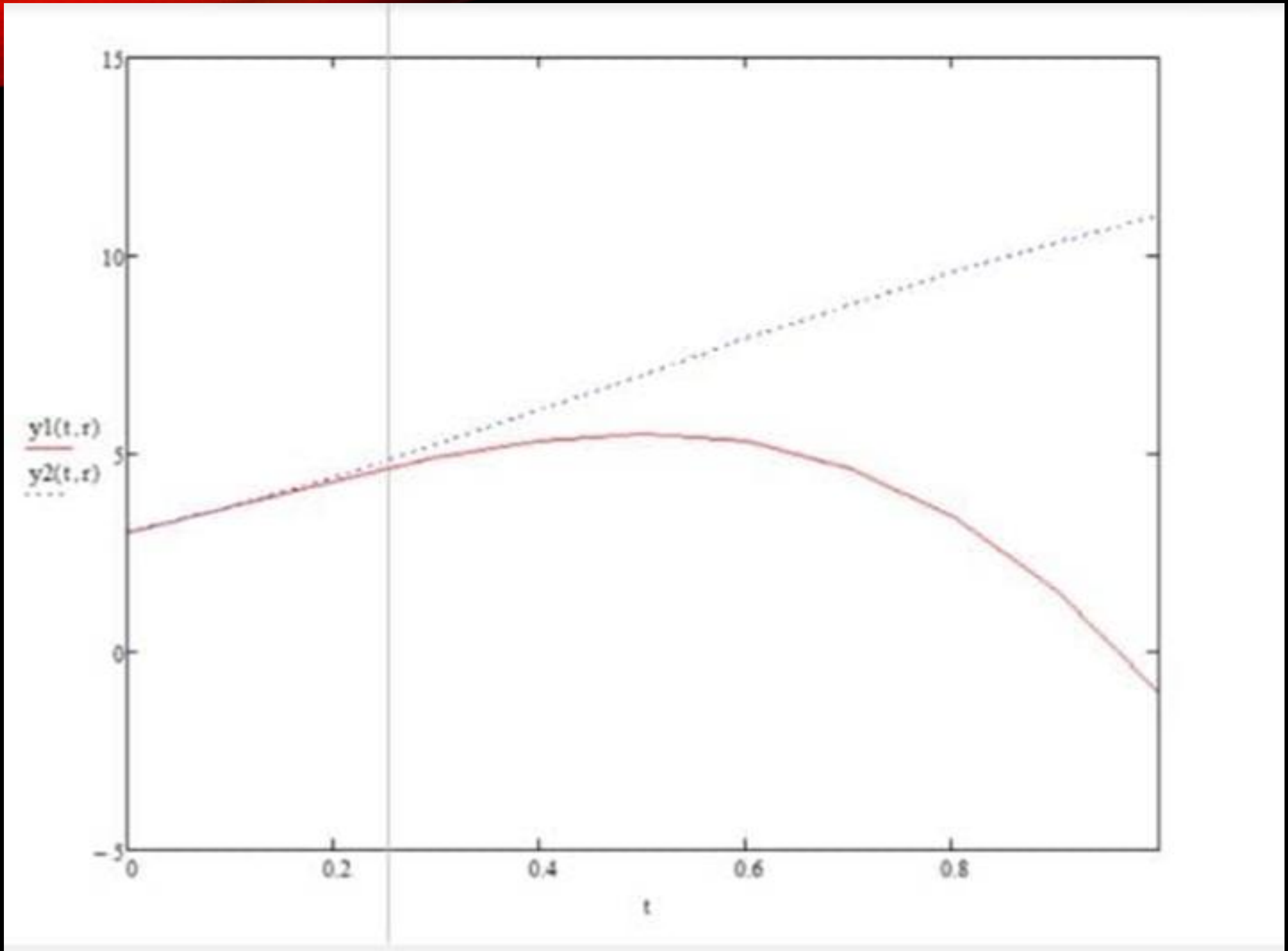
When r=0.1



When r=0.5



When r=1



Example 1: Consider the following second-order fuzzy linear differential equation . [4]

$$\begin{cases} y'' - 4y' + 4y = 0\ , t \in [0,1] \\ \quad \tilde{y}\,(0) = (2 + r, 4 - r) \\ \quad \tilde{y}'\,(0) = (5 + r, 7 - r) \end{cases} \qquad \text{............(27)}$$

The exact solution is as follows :

$$\underline{Y}(t, r) = (2 + r)e^{2t} + (1 - r)te^{2t}$$

$$\overline{Y}(t, r) = (4 - r)e^{2t} + (r - 1)te^{2t}$$

To apply the $\mathcal{L}(V|M)$,first we rewrite Eq (19)in the form

$$\underline{\lambda}(t) = \underline{\lambda}(t - s)$$

$$\overline{\lambda}(t) = \overline{\lambda}(t - s)$$

$$\mathcal{L}\left\{\underline{y}_{n+1}(t,r)\right\} = \mathcal{L}\left\{\underline{y}_n(t,r)\right\} + \mathcal{L}\left\{\int_0^t \underline{\lambda}(t-s)\left[\frac{d^2}{ds^2}\underline{y}_n - 4\frac{d}{ds}\overline{y}_n + 4\underline{y}_n\right]ds\right\}$$

$$\mathcal{L}\{\overline{y}_{n+1}(t,r)\} = \mathcal{L}\{\overline{y}_n(t,r)\} + \mathcal{L}\left\{\int_0^t \overline{\lambda}(t-s)\left[\frac{d^2}{ds^2}\overline{y}_n - 4\frac{d}{ds}\underline{y}_n + 4\overline{y}_n\right]ds\right\}$$

$$\mathcal{L}\left\{\underline{y}_{n+1}(t,r)\right\} = \mathcal{L}\left\{\underline{y}_n(t,r)\right\} + \mathcal{L}\left\{\underline{\lambda}(t)\left[\frac{d^2}{dt^2}\underline{y}_n - 4\frac{d}{dt}\overline{y}_n + 4\underline{y}_n\right]\right\}$$

$$\mathcal{L}\{\overline{y}_{n+1}(t,r)\} = \mathcal{L}\{\overline{y}_n(t,r)\} + \mathcal{L}\left\{\overline{\lambda}(t)\,[\frac{d^2}{dt^2}\overline{y}_n - 4\frac{d}{dt}\underline{y}_n + 4\overline{y}_n\right\}$$

$$\mathcal{L}\left\{\underline{y}_{n+1}(t,r)\right\}$$
$$= \mathcal{L}\left\{\underline{y}_n(t,r)\right\} + \mathcal{L}\{\underline{\lambda}(t)\}\, s^2\mathcal{L}\left\{\underline{y}_n\right\} - s\,\underline{y}_n(0) - \underline{y}_n'(0) - 4\left(s\,\mathcal{L}\left\{\underline{y}_n\right\} - \underline{y}(0)\right)$$
$$+ 4\,\mathcal{L}\left\{\underline{y}_n\right\}$$

$$\mathcal{L}\left\{\underline{y}_{n+1}(t,r)\right\} = \mathcal{L}\left\{\underline{y}_n(t,r)\right\} + \mathcal{L}\{\underline{\lambda}(t)\}[\,(s^2 - 4s + 4)\mathcal{L}\left\{\underline{y}_n\right\} - s\,\underline{y}_n(0) - \underline{y}_n'(0)\,]$$

Take the first variation resect to $y_n(t)$ thus

$$\mathcal{L}\left\{s\,\underline{y}_{n+1}\right\} = \left(1 + \mathcal{L}\{\underline{\lambda}(t)\}(s^2 - 4s + 4)\right)\mathcal{L}\left\{s\underline{y}_n\right\}$$

$$1 + \mathcal{L}\{\underline{\lambda}(t)\}(s^2 - 4s + 4) = 0$$

$$\mathcal{L}\{\underline{\lambda}(t)\} = \frac{-1}{s^2 - 4s + 4} = \frac{-1}{(s-2)(s-2)} = \frac{-1}{(s-2)^2}$$

$$\underline{\lambda}(t) = -te^{2t}$$

$$\underline{\lambda}(t) = \overline{\lambda}(t)$$

$$\mathcal{L}\left\{\underline{y}_{n+1}\right\} = \mathcal{L}\left\{\underline{y}_n\right\} + \mathcal{L}\{-te^{2t}\}\,\mathcal{L}\left\{\frac{d^2}{dt^2}\underline{y}_n - 4\frac{d}{dt}\overline{y}_n + 4\,\underline{y}_n\right\}$$

$$\mathcal{L}\{\overline{y}_{n+1}\} = \mathcal{L}\{\overline{y}_n\} + \mathcal{L}\{-te^{2t}\}\,\mathcal{L}\left\{\frac{d^2}{dt^2}\overline{y}_n - 4\frac{d}{dt}\underline{y}_n + 4\,\overline{y}_n\right\}$$

Let $\underline{y}_\circ = (2 + r) + t(5 + r)$

$$\overline{y}_\circ = (4 - r) + t(7 - r)$$

$$\mathcal{L}\{\underline{y}_1\} = \mathcal{L}\{\underline{y}_\circ\} + \mathcal{L}\{-te^{2t}\}\,\mathcal{L}\left\{\frac{d^2}{dt^2}\underline{y}_\circ - 4\frac{d}{dt}\,\underline{y}_\circ + 4\underline{y}_\circ\right\}$$

$$\mathcal{L}\{\overline{y}_1\} = \mathcal{L}\{\overline{y}_\circ\} + \mathcal{L}\{-te^{2t}\}\,\mathcal{L}\left\{\frac{d^2}{dt^2}\overline{y}_\circ - 4\frac{d}{dt}\,\overline{y}_\circ + 4\,\overline{y}_\circ\right\}$$

Invers Laplace transform yields ,

$$\underline{y}_1 = (2+r) + (5+r)t + (3 + 6te^{2t} - 3e^{2t}) - (\tfrac{1}{4}t + \tfrac{1}{4} + \tfrac{1}{4}te^{2t} - \tfrac{1}{4}e^{2t})$$

$$\overline{y}_1 = (4-r) + (7-r)t + (3 + 6te^{2t} - 3e^{2t}) - (28-4r)(\frac{1}{4}t + \frac{1}{4}te^{2t} - \frac{1}{4}e^{2t})$$

With are the exact solution of this problem, i.e , the exact solution is obtained in only one iteration

Example 2:

In order to solve the FIVP:

$$'(x) = x \quad (x),\ x \in [-1, 1]$$

with initial condition in parametric form:

$$(-1) = (\quad _{r0}, \quad _{r0})$$

$$= (\quad -0.5(1-r), \quad +0.5(1-r))$$

The method of solution will be discussed for $x > 0$ and then for $x \leq 0$ as the following cases show:

a. If x < 0: The parametric form in this case:

$$'(x;r) = x \quad (x;r) \quad \text{and} \quad '(x;r) = x \quad (x;r)$$

with initial conditions are:

$\underline{\tilde{y}}(-1; r) = \underline{\tilde{y}}_{r0}$ and $\overline{\tilde{y}}(-1; r) = \ _{r0}$

$$\text{let } t = x + 1$$

$$x = t - 1$$

$$\underline{y}' = (t-1)\overline{y} \; , \overline{y}' = (t-1)\underline{y}$$

$$\mathcal{L}\left\{\underline{y_{n+1}}\right\} = \left\{\underline{y_n}\right\}\mathcal{L} + \mathcal{L}\{-1\}.\mathcal{L}\{\underline{y'_n} - (t-1)\overline{y_n}\}$$

$$\mathcal{L}\{\overline{y_{n+1}}\} = \mathcal{L}\{\overline{y_n}\} + \mathcal{L}\{-1\}.\mathcal{L}\left\{\overline{y'_n} - (t-1)\underline{y_n}\right\}$$

$$\underline{y_0} = \sqrt{e} - 0.5(1-r) \; , \overline{y_0} = \sqrt{e} + 0.5(1-r)$$

$$\underline{y_1} = \underline{y_0} + \left(\sqrt{e} + 0.5(1-r)\right)\frac{(x+1)^2}{2} - \sqrt{e} + 0.5(1-r)(x+1)$$

$$\overline{y_1} = \overline{y_0} + \left(\sqrt{e} - 0.5(1-r)\right)\frac{(x+1)^2}{2} - \sqrt{e} - 0.5(1-r)(x+1)$$

b. If x ≥ 0: The parametric are given by:

$\underline{\tilde{y}}'(x; r) = x\,\tilde{y}(x; r)$ and $'(x; r) = x \quad (x; r)$

with initial conditions are r0, r0.

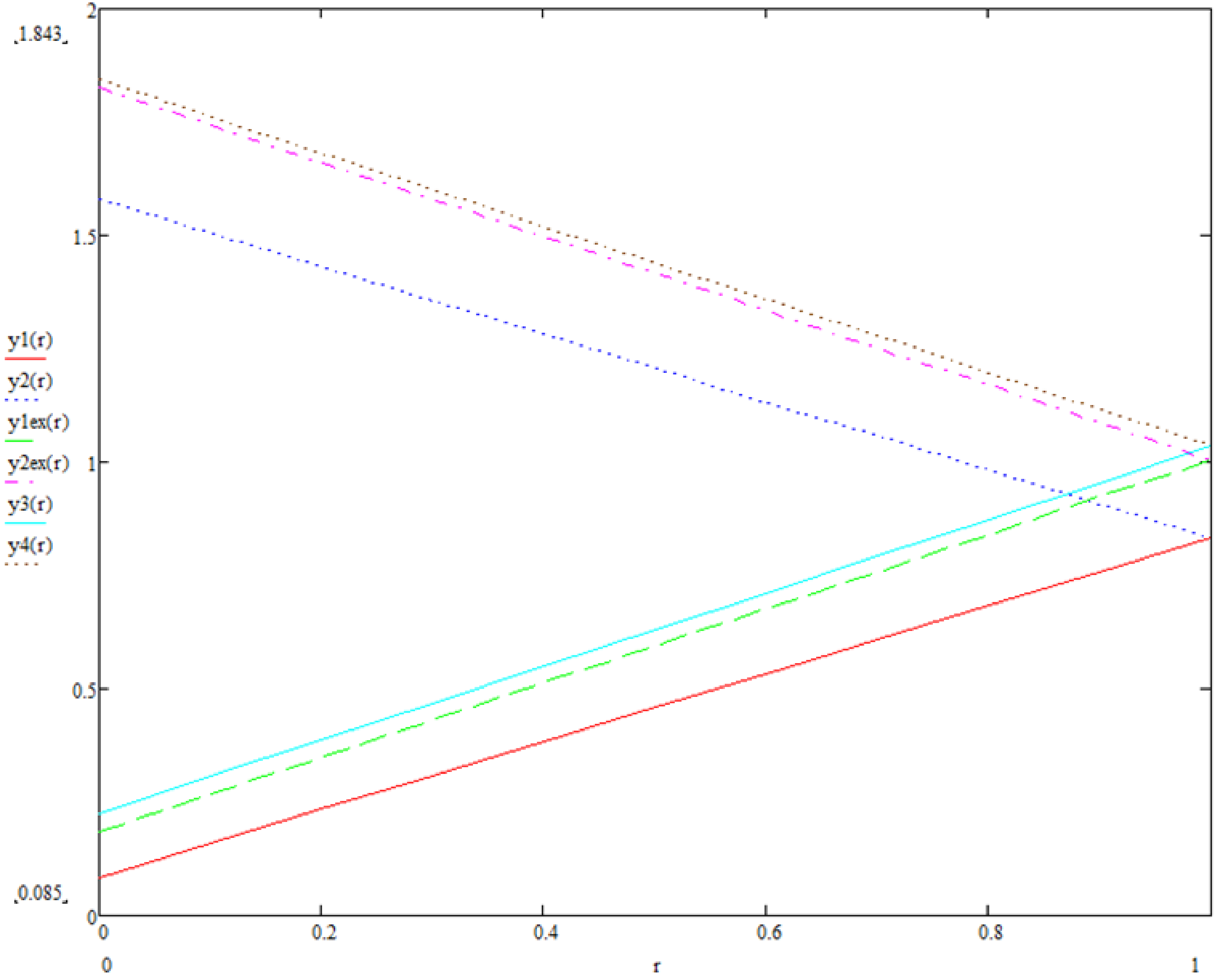
$$\mathcal{L}\{\underline{y_{n+1}}\} = \{\underline{y_n}\}\mathcal{L} + \mathcal{L}\{-1\}.\mathcal{L}\{\underline{y'_n} - x\underline{y_n}\}$$

$$\mathcal{L}\{\overline{y_{n+1}}\} = \mathcal{L}\{\overline{y_n}\} + \mathcal{L}\{-1\}.\mathcal{L}\{\overline{y'_n} - x\overline{y_n}\}$$

$$\underline{y_0} = \sqrt{e} - 0.5(1-r)\,, \overline{y_0} = \sqrt{e} + 0.5(1-r)$$

$$\underline{y_1} = \underline{y_0} + \left(\sqrt{e} - 0.5(1-r)\right)\frac{x^2}{2}$$

$$\overline{y_1} = \overline{y_0} + \left(\sqrt{e} + 0.5(1-r)\right)\frac{x^2}{2}$$



1.843
2
1.5
1
0.5
0.085
0
y1(r)
y2(r)
y1ex(r)
y2ex(r)
y3(r)
y4(r)
0
0.2
0.4
0.6
0.8
0
r
1

Thank you