5 Some Applications of Group Theory

5.1 Cayley Theorem

Theorem(5-1-1): (Cayley Theorem)

Every group is an isomorphic to a group of permutations.

This means if $(G,*)$ is any group, then $(G,*) \cong (F_G,\circ)$, where $F_G = \{f_a : a \in G\}, f_a : G \longrightarrow G \ni f_a(x) = a * x, \forall x \in G$.

Proof: define $g: G \longrightarrow F_G$ by $g(a) = f_a, \forall a \in G$

To prove g is a homomorphism, one to one and onto.

1. g is a homomorphism, let $a, b \in G$

$g(a * b) = f_{a*b} = f_a \circ f_b = g(a) \circ g(b) \Longrightarrow g$ is a homomorphism.

2. g is a one to one, let $g(a) = g(b),\ \forall a, b \in G$

$\Longrightarrow f_a = f_b \Longrightarrow f_a(x) = f_b(x) \Longrightarrow a * x = b * x \Longrightarrow a = b$

$\Longrightarrow g$ is a one to one.

3. g is a onto, $g(G) = \{g(a): a \in G\} = \{f_a: a \in G\} = F_G$

Therefore, $G \cong F_G$ ■

Corollary(5-1-2):

Every finite group $(G,*)$ of order n is an isomorphic to $(S_n,\circ)$.

Example(5-1-3):

Consider the following Cayley table of a group $(G = \{e,a,b,c\},*)$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Show that$(G,*)$ is an isomorphic to a subgroup of $(S_4,\circ)$.

Solution:

$f_e = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$, $\quad f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} =$

$(1)(2)(3)(4) = (1)$

$$f_a = \begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34)$$

$$f_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$$

$$f_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}, \quad f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$$

Hence, $(G,*)$ is an isomorphic to the subgroup of $(S_4,\circ)$:

$\{(1),(12)(34),(13)(24),(14)(23)\}$.

<u>Example(5-1-4):</u> (Homework)

Let $(G = \{1,-1,i,-i\},\cdot)$ be a group, apply Cayley Theorem on G.

<u>Example(5-1-5):</u> (Homework)

Show that $(Z_3,+_3)$ is an isomorphic to a subgroup of $(S_3,\circ)$.

<u>Exercises(5-1-6):</u>

- Apply Cayley Theorem on $(Z_4,+_4)$.
- Apply Cayley Theorem on $(G = \{\pm 1, \pm i, \pm j, \pm k\},\cdot)$.
- Apply Cayley Theorem on $(G = \{1,-1\},\cdot)$.

- Apply Cayley Theorem on $(\mathrm{G} = \{\mathrm{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, D =, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \cdot)$.

5.2 Direct Product

Definition(5-2-1):

Let $(H,*)$ and $(K,*)$ be two normal subgroups of $(G,*)$, then $(G,*)$ is called an internal direct product of H and K (G is a decomposition by H and K) if and only if $G = H * K$ and $H \cap K = \{e\}$.

Example(5-2-2):

Consider the following Cayley table of a group $(G = \{e, a, b, c\},*)$, $a^2 = b^2 = c^2 = e$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $H = \{e, a\}$ and $K = \{e, b\}$, show that $G = H \otimes K$ is a decomposition by H and K.

Solution: $H, K \Delta G$ since G is a commutative group

$H * K = \{e, a, b, c\}$ and $H \cap K = \{e\}$

Hence, $G = H \otimes K$ is decomposition by H and K.

Example(5-2-3):

Let $(G,*)$ be any group with $H = G$ and $K = \{e\}$, show that

$G = H \otimes K$ is a decomposition by H and K.

Solution: $H, K \Delta G$

$H * K = G * \{e\} = G$

$H \cap K = G \cap \{e\} = \{e\}$

Therefore, $G = H \otimes K$ is a decomposition by H and K.

Example(5-2-4):

Let $(Z_4, +_4)$ be a group. Is Z_4 has a proper decomposition.

Solution: the subgroups of Z_4 are $Z_4, \{0,2\}, \{0\}$

Let $H = Z_4$ and $K = \{0,2\}$

$H \otimes_4 K = Z_4 \otimes_4 \{0,2\} = Z_4$

$H \cap K = Z_4 \cap \{0,2\} = \{0,2\}$

So, $Z_4 \neq Z_4 \otimes \{0,2\}$

Let $H = \{0\}$ and $K = \{0,2\}$

$H \otimes_4 K = K \neq Z_4$

Therefore, Z_4 has no proper decomposition.

Theorem(5-2-5):

Let H and K be two subgroups of G and $G = H \otimes K$, then ${}^{G}/_{H} \cong K$ and ${}^{G}/_{K} \cong H$.

Proof:

Since $G = H \otimes K \Longrightarrow H * K = G$ and $H \cap K = \{e\}$

${}^{G}/_{H} = {}^{H*K}/_{H}$ and ${}^{H*K}/_{H} \cong {}^{K}/_{H \cap K}$ (by second theorem of isomorphic)

${}^{G}/_{H} \cong {}^{K}/_{\{e\}} \Longrightarrow {}^{G}/_{H} \cong K$ and

${}^{G}/_{K} = {}^{H*K}/_{K}$ and ${}^{H*K}/_{K} \cong {}^{H}/_{H \cap K}$

${}^{G}/_{K} \cong {}^{H}/_{\{e\}} \Longrightarrow {}^{G}/_{K} \cong H$ ■

Definition(5-2-6):

Let $(G_1,*)$ and $(G_2,\circ)$ be two groups, define $G_1 \times G_2 = \{(a,b): a \in G_1, b \in G_2\}$ such that $(a,b)\odot(c,d) = (a * c, b \circ d) \ni a,c \in G_1, b,d \in G_2$. Then $(G_1 \times G_2, \odot)$ is a group which is called an external direct product of G_1 and G_2.

Example(5-2-7): (Homework)

Show that $(G_1 \times G_2, \odot)$ is a group.

Example(5-2-8):

Let $G_1 = (\mathrm{Z}_3, +_3)$ and $G_2 = (\mathrm{Z}_2, +_2)$. Find $G_1 \times G_2$.

Solution:

$G_1 \times G_2 = \mathrm{Z}_3 \times \mathrm{Z}_2$

$= \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}$

$(1,1)\odot(2,1) = (0,0)$

$\mathrm{o}(\mathrm{Z}_3 \times \mathrm{Z}_2) = o(\mathrm{Z}_3).o(\mathrm{Z}_2) = 6.$

Theorem(5-2-9):

Let $(G_1,*)$ and $(G_2,\circ)$ be two groups, then

1. $(G_1 \times G_2, \odot)$ is an abelian if and only if both G_1 and G_2 are abelian.
2. $G_1 \times \{e_2\} \vartriangle G_1 \times G_2$.
3. $\{e_1\} \times G_2 \vartriangle G_1 \times G_2$.
4. $G_1 \cong G_1 \times \{e_2\}$.
5. $G_2 \cong \{e_2\} \times G_2$.

Proof:

1. $(\Longrightarrow)$ suppose that $G_1 \times G_2$ is an abelian, to prove $G_1$and G_2 are abelian.

Let $(a, e_2), (b, e_2) \in G_1 \times G_2 \ni a, b \in G_1, e_2 \in G_2$

Since $G_1 \times G_2$ is an abelian, then

$(a, e_2) \odot (b, e_2) = (b, e_2) \odot (a, e_2)$

$(a * b, e_2) = (b * a, e_2) \Longrightarrow a * b = b * a$

Hence, $(G_1,*)$ is an abelian.

Similarly that $(G_2,*)$ is an abelian.

($\Leftarrow$) suppose that $(G_1,*)$ and $(G_2,\circ)$ are abelian, to prove $G_1 \times G_2$ is an abelian.

Let $(a,b),(c,d) \in G_1 \times G_2$, to prove $(a,b)\odot(c,d) = (c,d)\odot(a,b)$

$(a,b)\odot(c,d) = (a*c, b*d)$

$(c,d)\odot(a,b) = (c*a, d*b)$

$a*c = c*a$ ($G_1$is an abelian)

$b*d = d*b$ ($G_2$is an abelian)

$\Longrightarrow (a,b)\odot(c,d) = (c,d)\odot(a,b)$

Therefore, $G_1 \times G_2$ is an abelian.

2. To prove $G_1 \times \{e_2\} \vartriangle G_1 \times G_2$

$G_1 \times \{e_2\} = \{(a,e_2): a \in G_1\} \neq \emptyset$

To prove $(G_1 \times \{e_2\},\odot)$ is a subgroup of $G_1 \times G_2$

Let $(a,e_2),(b,e_2) \in G_1 \times \{e_2\}$

$(a,e_2)\odot(b,e_2)^{-1} = (a,e_2)\odot(b^{-1},{e_2}^{-1}) = (a*b^{-1},e_2)$

So, $(G_1 \times \{e_2\},\odot)$ is a subgroup of $G_1 \times G_2$.

To prove $G_1 \times \{e_2\} \triangle G_1 \times G_2$

Let $(x, y) \in G_1 \times G_2$ and $(a, e_2) \in G_1 \times \{e_2\}$

To prove $(x, y) \odot (a, e_2) \odot (x, y)^{-1} \in G_1 \times \{e_2\}$

$(x * a * x^{-1}, y * e_2 * y^{-1}) = (x * a * x^{-1}, e_2) \in G_1 \times \{e_2\}$

Hence, $G_1 \times \{e_2\} \triangle G_1 \times G_2$.

3. (**Homework).**
4. To prove $G_1 \cong G_1 \times \{e_2\}$.

Proof:

Define $f: (G_1, *) \longrightarrow (G_1 \times \{e_2\}, \odot) \ni f(a) = (a, e_2)$

f is a map ? let $a_1, a_2 \in G_1$ and $a_1 = a_2 \Longrightarrow (a_1, e_2) = (a_2, e_2) \Longrightarrow f(a_1) = f(a_2)$, so f is a map

f is an one to one ? let $f(a_1) = f(a_2) \Longrightarrow (a_1, e_2) = (a_2, e_2) \Longrightarrow a_1 = a_2$, so f is a one to one.

f is a homomorphism ? $f(a * b) = (a * b, e_2) = (a, e_2) \odot (b, e_2) = f(a) \odot f(b)$, so f is a homomorphism

f is an onto ? $R_f = \{f(a): a \in G_1\} = \{(a, e_2): a \in G_1\} = G_1 \times \{e_2\}$ so f is an onto.

Therefore, $(G_1,*) \cong (G_1 \times \{e_2\}, \odot)$ ■

5. **(Homework)**

Theorem(5-2-10):

Let $(G_1,*)$ and $(G_2,\circ)$ be two p-groups, then $(G_1 \times G_2, \odot)$ is a p-group.

Proof:

Since G_1 is p-group $\Longrightarrow o(G_1) = p^{k_1}, k_1 \in Z^+$

Since G_2 is p-group $\Longrightarrow o(G_2) = p^{k_2}, k_2 \in Z^+$

$$o(G_1 \times G_2) = o(G_2) \times o(G_1) = p^{k_1} \times p^{k_2}$$
$$= p^{k_1+k_2}, k_1 + k_2 \in Z^+$$

Therefore, $G_1 \times G_2$ is a p-group ■

Exercises(5-2-11):

- Let $H = \{0,2,4\}$ and $K = \{0,3\}$ are subgroups of $(\mathrm{Z}_6, +_6)$, show that $\mathrm{Z}_6 = H \otimes K$ is a decomposition.
- Let $H = \{0\}$, show that $\mathrm{Z}_7 = H \otimes \mathrm{Z}_7$ is a decomposition.
- Find $\mathrm{Z}_3 \times \mathrm{Z}_7$.

- Is $S_3 \times Z_2$ an abelian?
- Is $G_s \times Z_2$ an abelian?
- Is $S_3 \times G_S$ an abelian?
- Is $\{\pm 1, \pm i\} \times Z_2$ an abelian?
- Is $Z_4 \times Z_8$ a p-group?
- Is $Z_5 \times Z_{25}$ a p-group?
- Is $Z_{11} \times Z_{121}$ a p-group?
- Is $Z_7 \times Z_{49}$ a p-group?
- Is $Z_{27} \times Z_3$ a p-group?
- Is $Z_5 \times Z_{125}$ a p-group?
- Is $Z_2 \times Z_{64}$ a p-group?
- Is $Z_4 \times Z_{128}$ a p-group?
- Is $Z_9 \times Z_{81}$ a p-group?
- Is $Z_{27} \times Z_{81}$ a p-group?
- Is $Z_{128} \times Z_8$ a p-group?
- Is $Z_2 \times Z_{256}$ a p-group?