Theorem(4-7):

Let $H \Delta G$ and both H, ${}^{G}/_{H}$ are solvable, then $(G,*)$ is a solvable.

Proof: since $(H,*)$ is a solvable $\Longrightarrow$

there is a finite collection of subgroups of $(G,*)$, $H_0, H_1, \dots, H_n$ such that

1. $G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$,
2. $H_{i+1} \Delta H_i \quad \forall i = 0, \dots, n-1$,
3. ${}^{H_i}/_{H_{i+1}}$ is a commutative group $\forall i = 0, \dots, n-1$.

Since $({}^{G}/_{H}, \otimes)$ is a solvable $\Longrightarrow$

there is a finite collection of subgroups of $(G,*)$, $\frac{G_0}{H}, \frac{G_1}{H}, \dots, \frac{G_r}{H}$ such that

1. $\frac{G}{H} = \frac{G_0}{H} \supset \frac{G_1}{H} \supset \cdots \supset \frac{G_r}{H} = \{e\} = H$,
2. $\frac{G_{i+1}}{H} \Delta \frac{G_i}{H} \quad \forall i = 0, \dots, r-1$,
3. $\left. \frac{G_i}{H} \middle/ \frac{G_{i+1}}{H} \right.$ is a commutative group $\forall i = 0, \dots, r-1$.

To prove $(G,*)$ is a solvable group.

$$\frac{G}{H} = \frac{G_0}{H} \Longrightarrow G = G_0$$

$$\frac{G_r}{H} = H \Longrightarrow G_r = \{e\} \text{ or } G_r = H$$

$$H \Delta G_r \Longrightarrow H \subseteq G_r \Longrightarrow G_r = H$$

So, there is a finite collection $G_0, G_1, \dots, G_r = H_0, H_1, \dots, H_n$ such that

1. $G = G_0 \supset G_1 \supset \cdots \supset G_r = H = H_0 \supset H_1 \supset \cdots \supset H_n = \{e\}$.
2. To prove $G_{i+1} \Delta G_i \quad \forall i = 0, \dots, r-1$

Let $x \in G_i$ and $a \in G_{i+1}$ to prove $x * a * x^{-1} \in G_{i+1}$

$$x \in G_i \Longrightarrow x * H \in \frac{G_i}{H}$$

$$a \in G_{i+1} \Longrightarrow a * H \in \frac{G_{i+1}}{H}$$

$$\frac{G_{i+1}}{H} \Delta \frac{G_i}{H} \Longrightarrow (x * H) \otimes (a * H) \otimes (x * H)^{-1} \in \frac{G_{i+1}}{H}$$

$$\Longrightarrow (x * a * x^{-1}) * H \in \frac{G_{i+1}}{H} \Longrightarrow x * a * x^{-1} \in G_{i+1}$$

$$\Longrightarrow G_{i+1} \Delta G_i$$

3. To prove $\frac{G_i}{G_{i+1}}$ is a commutative group $\forall i = 0, \ldots, r - 1$

$\frac{\frac{G_i}{H}}{\frac{G_{i+1}}{H}}$ is a commutative group and $\frac{\frac{G_i}{H}}{\frac{G_{i+1}}{H}} \cong \frac{G_i}{G_{i+1}}$ $(\frac{\frac{G}{H}}{\frac{K}{H}} \cong \frac{G}{K})$

$\Longrightarrow \frac{G_i}{G_{i+1}}$ is a commutative group

Therefore, $(G,*)$ is a solvable group ■

Exercises(4-8);

- Show that every p-group is a solvable group.
- Show that $(S_4, \circ)$ is a solvable group.
- Show that $(Z_4, +_4)$ is a solvable group.
- Show that $(Z_8, +_8)$ is a solvable group.
- Show that $(Z_5, +_5)$ is a solvable group.
- Show that $(Z_6, +_6)$ is a solvable group.
- Show that $(Z_{12}, +_{12})$ is a solvable group.
- Show that $(Z_{24}, +_{24})$ is a solvable group.