Remark(3-12):

The group G has exactly one sylow p-subgroup H if and only if $H \Delta G$.

Example(3-13):

$(S_3, \circ), H = \{f_1 = i, f_2 = (123), f_3 = (132)\}$

$H \Delta G \Longrightarrow$ H is a sylow 3-subgroup of S_3,

So, there is one sylow 3-subgroup of S_3.

Exercises(3-14);

- Show that there is no simple group of order 200.
- Show that there is no simple group of order 56.
- Show that there is no simple group of order 20.
- Show that whether $(G_\ell, \cdot)$ is a sylow.

1. **Solvable Groups and Their Applications**

Definition(4-1):

A group $(G,*)$ is called a solvable group if and only if, there is a finite collection of subgroups of $(G,*)$, $H_0, H_1, \dots, H_n$ such that

1. $G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$,
2. $H_{i+1} \Delta H_i \quad \forall i = 0, \dots, n-1$,
3. ${}^{H_i}/_{H_{i+1}}$ is a commutative group $\forall i = 0, \dots, n-1$.

Example(4-2):

Show that, every commutative group is a solvable group.

Solution:

Suppose that $(G,*)$ is a commutative, to show that $(G,*)$ is a solvable.

Let $G = H_0$ and $H_1 = \{e\}$

1. $G = H_0 \supset H_1 = \{e\}$
2. $H_1 \Delta H_0$ satisfies, since $\{e\} \Delta G$, or (every subgroup of commutative group is a normal)
3. ${}^{G}/_{\{e\}} \cong G$ is a commutative group, or (the quotient of commutative group is a commutative)

So, $(G,*)$ is a solvable group,

Example(4-3):

Show that $(S_3,\circ)$ is a solvable group.

Solution: let $H_0 = S_3, H_1 = \{f_1 = i, f_2 = (123), f_3 = (132)\}, H_2 = \{f_1\}$

1. $S_3 = H_0 \supset H_1 \supset H_2 = \{e\}$
2. $H_2 \Delta H_1$ satisfies, since $\{ f_1\}\Delta\{f_1, f_2, f_3\}$, $H_1 \Delta H_0$ is true, since $[S_3 : H_1] = 2 \Longrightarrow H_1 \Delta S_3$
3. To prove ${}^{H_i}/_{H_{i+1}}$ is a commutative group $\forall i = 0,1$

$o\left({}^{H_1}/_{H_2}\right) = \frac{o(H_1)}{o(H_2)} = \frac{3}{1} = 3 < 6 \Longrightarrow {}^{H_1}/_{H_2}$ is a commutative group

$o\left({}^{H_0}/_{H_1}\right) = \frac{o(H_0)}{o(H_1)} = \frac{6}{3} = 2 < 6 \Longrightarrow {}^{H_0}/_{H_1}$ is a commutative group

Therefore, $(S_3, \circ)$ is a solvable group.

Example(4-4): (Homework)

Show that $(G_s, \circ)$ is a solvable group.