Examples(2-8):

Apply theorem(2-7) on $(\mathrm{Z}_{32}, +_{32})$.

Solution:

$|\mathrm{Z}_{32}| = 32 = 2^5$ is a 2- group.

By theorem (2-7), H and ${}^{\mathrm{G}}/_{H}$ are 2- groups.

${}^{\mathrm{o(G)}}/_{o(H)} \Rightarrow o(H) = 2^x, 0 \leq x \leq 5.$

$o(H) = 2^0$ or 2^1 or 2^2 or 2^3 or 2^4 or 2^5,

$o(H) = 2^0$ is a 2- group $\Rightarrow o\left({}^{\mathrm{G}}/_{H}\right) = {}^{\mathrm{o(G)}}/_{o(H)} = \frac{2^5}{2^0} =$ 2^5 is a 2- group.

$o(H) = 2^1$ is a 2- group $\Rightarrow {}^{\mathrm{o(G)}}/_{o(H)} = 2^4$

$o(H) = 2^2$ is a 2- group $\Rightarrow {}^{\mathrm{o(G)}}/_{o(H)} = 2^3$

$o(H) = 2^3$ is a 2- group $\Rightarrow {}^{\mathrm{o(G)}}/_{o(H)} = 2^2$

$o(H) = 2^4$ is a 2- group $\Rightarrow {}^{\mathrm{o(G)}}/_{o(H)} = 2$

$o(H) = 2^5$ is a 2- group $\Longrightarrow {}^{\mathrm{o(G)}}\!/_{o(H)} = 1$.

Remark(2-9);

If G is a non-trivial p- group, then $\mathrm{Cent(G)} \neq e$.

Theorem(2-10):

Every group of order p^2 is an abelian.

Proof: Let G be a group of order p^2, to prove G is an abelian.

Let Cent(G) is a subgroup of G.

By Lagrange Theorem ${}^{\mathrm{o(G)}}\!/_{o(\mathrm{Cent(G)})}$,

$\Longrightarrow {}^{\mathrm{p}^2}\!/_{o(\mathrm{Cent(G)})}$

$\Longrightarrow o\big(\mathrm{Cent(G)}\big) = \mathrm{p}^0$ or p^1 or p^2

If $o\big(\mathrm{Cent(G)}\big) = \mathrm{p}^0 \Longrightarrow o\big(\mathrm{Cent(G)}\big) = \{e\}$, but this is contradiction with remark(2-9), so $o\big(\mathrm{Cent(G)}\big) \neq \mathrm{p}^0$.

If $o\big(\mathrm{Cent(G)}\big) = \mathrm{p}^2 = o(G) \Longrightarrow \mathrm{Cent(G)} = G$

$\Longrightarrow G$ is an abelian.

If $o(\mathrm{Cent(G)}) = \mathrm{p}^1 \Longrightarrow o\left({}^{G}/_{\mathrm{Cent(G)}}\right) = \frac{\mathrm{p}^2}{\mathrm{p}^1} = \mathrm{p}$

${}^{G}/_{\mathrm{Cent(G)}}$ is a cyclic.

Therefore, G is an abelian ■

Remark(2-11):

The converse of theorem(2-10) is not true in general, for example $(\mathrm{Z}_8, +_8)$ is an abelian, but $o((\mathrm{Z}_8) = 2^3 \neq p^2$.