1. P- Groups and Related Concepts.

<u>Definition(2-1):</u> (**p- Group**)

A finite group $(G,*)$ is said to be p- $group$ if and only if the order of each element of G is a power of fixed prime p.

<u>Definition(2-2):</u> (**p- Group**)

A finite group $(G,*)$ is said to be p- $group$ if and only if $|G| = p^k, k \in \mathrm{Z}$, where p is a prime number.

<u>Example(2-3):</u>

Show that $(\mathrm{Z}_4, +_4)$ is a p- group.

<u>Solution:</u> $\mathrm{Z}_4 = \{0,1,2,3\}$ and $|\ \mathrm{Z}_4| = 4 = 2^2$

$\Rightarrow$ Z_4 is a 2- group, with

$\mathrm{o}(0) = 1 = 2^0$,

$\mathrm{o}(1) = 4 = 2^2$,

$\mathrm{o}(2) = 2 = 2^1$,

$\mathrm{o}(3) = 4 = 2^2$.

Example(2-4):

Determine whether $(\mathrm{Z}_6, +_6)$ is a p- group.

Solution: $\mathrm{Z}_6 = \{0,1,2,3,4,5\}$ and $|\ \mathrm{Z}_6| = 6 \neq P^k$

$\Rightarrow$ Z_6 is not p- group.

Example(2-5): (Homework)

Determine whether $(\mathrm{G}_s, \circ)$ is a p- group.

Examples(2-6):

- $(\mathrm{Z}_8, +_8)$ is a 2- group, since $|\mathrm{Z}_8| = 8 = 2^3$,
- $(\mathrm{Z}_9, +_9)$ is a 3- group, since $|\mathrm{Z}_9| = 9 = 3^2$,
- $(\mathrm{Z}_{25}, +_{25})$ is a 5- group, since $|\mathrm{Z}_{25}| = 25 = 5^2$.

Theorem(2-7):

Let $\mathrm{H}\Delta\mathrm{G}$, then G is a p- group if and only if H and ${}^{\mathrm{G}}/_{H}$ are p- groups.

Proof: $(\Longrightarrow)$ Assume that G is a p- group, to prove that H and ${}^{\mathrm{G}}/_{H}$ are p- groups.

Since G is a p- group $\Longrightarrow$ $o(a) = p^x$, for some $x \in Z^+, \forall a \in G$.

Since $H \subseteq G$ $\Longrightarrow \forall a \in H$ group $\Longrightarrow$ $o(a) = p^x$, for some $x \in Z^+$.

So, H is a p- group.

To prove $G/_H$ is a p- group.

Let $a * H \in G/_H$, to prove $o(a * H)$ is a power of p.

$(a * H)^{p^x} = a^{p^x} * H = e * H = H$, ($a^{p^x} = e$ since G is a p- group$\Longrightarrow$ $o(a) = p^x$)

($\Longleftarrow$) Suppose that H and $G/_H$ are p- groups, to prove G is a p- group.

Let $a \in H$, to prove $o(a)$ is a power of p.

$(a * H)^{p^x} = H \ldots (1)$ ($G/_H$ is a p- group)

$(a * H)^{p^x} = a^{p^x} * H \ldots (2)$

From (1) and (2), we have $a^{p^x} * H = H \Longrightarrow a^{p^x} \in H$ and H is a p- group,

$$\Longrightarrow o\left(a^{p^x}\right) = p^r, r \in Z^+$$

$$\Longrightarrow \left(a^{p^x}\right)^{p^r} = e \Longrightarrow a^{p^{x+r}} = e,\ x + r \in Z^+,$$

$$\Longrightarrow o(a) = p^{x+r}$$

Therefore, G is a p- group ■