Example(1-13):

The group $(\mathrm{Z},+)$ has no a composition chain, since the normal subgroups of $(\mathrm{Z},+)$ are the cyclic subgroups $(\langle \mathrm{n} \rangle),+)$, n a nonnegative integer, Since the inclusion $\langle \mathrm{kn} \rangle \subseteq \langle n \rangle$ holds for all $\mathrm{k} \in \mathrm{Z}_{+}$, there always exists a proper subgroup of any given group.

Definition(1-14):

A normal subgroup $(H,*)$ is called a *maximal normal subgroup* of the group $(G,*)$ if $H \neq G$ and there exists no normal subgroup $(K,*)$ of $(G,*)$ such that $H \subset K \subset G$.

Example(1-15):

In the group $(\mathrm{Z}_{24},+_{24})$, the cyclic subgroups $(\langle 2 \rangle,+_{24})$ and $(\langle 3 \rangle,+_{24})$ are both maximal normal with orders 12 and 8, respectively.

Example(1-16):

Determine the maximal normal subgroups in the group $(\mathrm{Z}_{12},+_{12})$.

Solution: The normal subgroups of $(\mathrm{Z}_{12}, +_{12})$ are:

$H_1 = (\langle 2 \rangle, +_{12}) = (\{0,2,4,6,8,10\}, +_{12})$

$H_2 = (\langle 3 \rangle, +_{12}) = (\{0,3,6,9\}, +_{12})$

$H_3 = (\langle 4 \rangle, +_{12}) = (\{0,4,8\}, +_{12})$

$H_4 = (\langle 6 \rangle, +_{12}) = (\{0,6\}, +_{12})$

The maximal normal subgroups of $(\mathrm{Z}_{12}, +_{12})$ are H_1 and H_2, since there is no normal subgroup in Z_{12} containing H_1 and H_2.

Remark(1-17):

A chain $G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$ is a composition of a group $(G,*)$, if each normal subgroup $(H_i,*)$ is a maximal normal subgroup of $(H_{i-1},*)$, for all $i = 1, \dots, n$.

Example(1-18);

In the group $(\mathrm{Z}_{12}, +_{12})$ the chains $\mathrm{Z}_{12} \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \{0\}$ is a composition of Z_{12} , since

$\langle 2 \rangle$ is a maximal normal subgroup of Z_{12},

$\langle 4\rangle$ is a maximal normal subgroup of $\langle 2\rangle$,

$\{0\}$ is a maximal normal subgroup of $\langle 4\rangle$, and

$Z_{12} \supset \langle 3\rangle \supset \langle 6\rangle \supset \{0\}$ is a composition of Z_{12}, since

$\langle 3\rangle$ is a maximal normal subgroup of Z_{12},

$\langle 6\rangle$ is a maximal normal subgroup of $\langle 3\rangle$,

$\{0\}$ is a maximal normal subgroup of $\langle 6\rangle$.

Theorem(1-19):

A normal subgroup $(H,*)$ of the group $(G,*)$ is a maximal if and only if the quotient $({}^{G}/_{H},\otimes)$ is a simple.

Proof:

$\Rightarrow$) Let K be a normal subgroup of G with $H \subseteq K$ there corresponds between $({}^{G}/_{H},\otimes)$ and $({}^{K}/_{H},\otimes)$ such that this correspondence is one-to-one. Hence, H is a maximal normal in K $\Rightarrow H$ is a maximal normal in G (by correspondence) $\Rightarrow {}^{G}/_{H}$ is a simple.

$\Leftarrow$) let ${}^{G}/_{H}$ be a simple

$\Rightarrow {}^{G}/_{H}$ has two normal subgroups which are $e * H$ and ${}^{G}/_{H}$, but $e * H = H$

Therefore H is a maximal ■