Example(1-10):

Find all chains in the following groups and determine their length and type.

- $(\mathrm{Z}_8, +_8)$;
- $(\mathrm{Z}_{12}, +_{12})$;
- $(\mathrm{Z}_{18}, +_{18})$ (**Homework**).

Solution: The subgroups of a group $(\mathrm{Z}_8, +_8)$ are :

$H_1 = (\mathrm{Z}_8, +_8)$

$H_2 = (\{0\}, +_8)$

$H_3 = (\langle 2\rangle, +_8) = (\{0,2,4,6\}, +_8)$

$H_4 = (\langle 4\rangle, +_8) = (\{0,4\}, +_8)$

Then the chains in $(\mathrm{Z}_8, +_8)$ are:

$\mathrm{Z}_8 \supset \{0\}$ is a trivial chain of length one.

$\mathrm{Z}_8 \supset \langle 2\rangle \supset \{0\}$ is a normal chain of length two, but it is not composition chain, since there is a normal subgroup $\langle 4\rangle$ in Z_8, such that $\langle 2\rangle \supset \langle 4\rangle$.

$Z_8 \supset \langle 4 \rangle \supset \{0\}$ is a normal chain of length two, but it is not composition chain, since there is a normal subgroup $\langle 2 \rangle$ in Z_8, such that $\langle 2 \rangle \supset \langle 4 \rangle$.

$Z_8 \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \{0\}$ is a composition chain of length three.

The subgroups of a group $(Z_{12}, +_{12})$ are :

$H_1 = (Z_{12}, +_{12})$

$H_2 = (\{0\}, +_{12})$

$H_3 = (\langle 2 \rangle, +_{12}) = (\{0,2,4,6,8,10\}, +_{12})$

$H_4 = (\langle 3 \rangle, +_{12}) = (\{0,3,6,9\}, +_{12})$

$H_5 = (\langle 4 \rangle, +_{12}) = (\{0,4,8\}, +_{12})$

$H_6 = (\langle 6 \rangle, +_{12}) = (\{0,6\}, +_{12})$

Then the chains in $(Z_{12}, +_{12})$ are:

$Z_{12} \supset \{0\}$ is a trivial chain of length one.

$Z_{12} \supset \langle 2 \rangle \supset \{0\}$ is a normal chain of length two.

$Z_{12} \supset \langle 3 \rangle \supset \{0\}$ is a normal chain of length two.

$Z_{12} \supset \langle 4 \rangle \supset \{0\}$ is a normal chain of length two.

$Z_{12} \supset \langle 6 \rangle \supset \{0\}$ is a normal chain of length two.

$Z_{12} \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \{0\}$ is a composition chain of length three.

$Z_{12} \supset \langle 3 \rangle \supset \langle 6 \rangle \supset \{0\}$ is a composition chain of length three.

Example(1-11):

Let $(G,*)$ be the group of symmetries of the square.

A normal chain for $(G,*)$ which fails to be a composition chain is

$$G \supset \{R_{180}, R_{360}\} \supset \{R_{360}\}.$$

Example(1-12): (Homework)

Determine the following chain whether normal, composition:

$$G \supset \{R_{90}, R_{180}, R_{270}, R_{360}\} \supset \{R_{180}, R_{360}\} \supset \{R_{360}\}.$$