Example(1-6):

Find all chains in the following groups and determine their length and type.

- $(\mathrm{Z}_6, +_6)$;
- $(\mathrm{Z}_8, +_8)$;
- $(\mathrm{Z}_{18}, +_{18})$ (**Homework**);
- $(\mathrm{Z}_{21}, +_{21})$ (**Homework).**

Solution: The subgroups of a group $(\mathrm{Z}_6, +_6)$ are :

$H_1 = (\mathrm{Z}_6, +_6)$

$H_2 = (\{0\}, +_6)$

$H_3 = (\langle 2\rangle, +_6) = (\{0,2,4\}, +_6)$

$H_4 = (\langle 3\rangle, +_6) = (\{0,3\}, +_6)$

Then the chains in $(\mathrm{Z}_6, +_6)$ are:

$\mathrm{Z}_6 \supset \{0\}$ is a trivial chain of length one

$\mathrm{Z}_6 \supset \langle 2\rangle \supset \{0\}$ is a normal chain of length two

$\mathrm{Z}_6 \supset \langle 3\rangle \supset \{0\}$ is a normal chain of length two.

The subgroups of a group $(Z_8, +_8)$ are :

$H_1 = (Z_8, +_8)$

$H_2 = (\{0\}, +_8)$

$H_3 = (\langle 2 \rangle, +_8) = (\{0,2,4,6\}, +_8)$

$H_4 = (\langle 4 \rangle, +_6) = (\{0,4\}, +_8)$

Then the chains in $(Z_8, +_8)$ are:

$Z_8 \supset \{0\}$ is a trivial chain of length one

$Z_8 \supset \langle 2 \rangle \supset \{0\}$ is a normal chain of length two

$Z_8 \supset \langle 4 \rangle \supset \{0\}$ is a normal chain of length two

$Z_8 \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \{0\}$ is a normal chain of length three.

Definition(1-7): (***Composition Chain***)

In the group $(G,*)$, the descending sequence of sets

$$G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$$

forms a *composition chain* for $(G,*)$ provided

1. $(H_i,*)$ is a subgroup of $(G,*)$,
2. $(H_i,*)$ is a normal subgroup of $(H_{i-1},*)$,

3. The inclusion $H_{i-1} \supseteq K \supseteq H_i$, where $(K,*)$ is a normal subgroup of $(H_{i-1},*)$, implies either $K = H_{i-1}$ or $K = H_i$.

Remark(1-8):

Every composition chain is a normal, but the converse is not true in general, the following example shows that.

Example(1-9):

In the group $(\mathrm{Z}_{24}, +_{24})$, the normal chain

$$\mathrm{Z}_{24} \supset \langle 2 \rangle \supset \langle 12 \rangle \supset \{0\}$$

is not a composition chain, since it may be further refined by inserting of the set $\langle 4 \rangle$ or $\langle 6 \rangle$. On other hand,

$$\mathrm{Z}_{24} \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \langle 8 \rangle \supset \{0\}$$

and

$$\mathrm{Z}_{24} \supset \langle 3 \rangle \supset \langle 6 \rangle \supset \langle 12 \rangle \supset \{0\}$$

are both composition chains for $(\mathrm{Z}_{24}, +_{24})$.