ESCI 341 – Atmospheric Thermodynamics
Lesson 7 – Vertical Structure

References: *An Introduction to Dynamic Meteorology*, Holton
Introduction to Theoretical Meteorology, Hess
Synoptic-dynamic Meteorology in Midlatitudes, Vol. 1, Bluestein
'An example of uncertainty in sea level pressure reduction', P.M. Pauley, *Mon. Wea. Rev.*, **13**, 1998, pp. 833-850

GEOPOTENTIAL

- **The acceleration due to gravity is not constant. It varies from place to place, with the largest variation due to latitude.**
  - **What we call gravity is the combination of the gravitational acceleration and the centrifugal acceleration from the Earth's rotation.**
  - **Gravity at the North Pole is approximately 9.83 m/s^2, while at the Equator it is about 9.78 m/s^2.**
- **Though small, the variation in gravity must be accounted for. We do this via the concept of *geopotential*.**
- **Geopotential is essentially the potential energy per unit mass.**
- **A surface of constant geopotential represents a surface along which all objects of the same mass have the same potential energy.**
- **If gravity were constant, a geopotential surface would lie at a constant altitude. Since gravity is not constant, a geopotential surface will have varying altitude.**
- ***Geopotential* is defined as**

$$\Phi \equiv \int_0^z g\,dz\,, \qquad \textbf{(1)}$$

or in differential form as

$$d\Phi = g\,dz\,.$$

- ***Geopotential height* is defined as**

$$Z \equiv \frac{\Phi}{g_0} = \frac{1}{g_0}\int_0^Z g\,dZ \qquad \textbf{(2)}$$

where g_0 is a constant called *standard gravity*, and has a value of 9.80665 m/s^2.
  - **Geopotential height is expressed in *geopotential meters*, abbreviated as gpm.**

- **If the change in gravity with height is ignored, geopotential height and geometric height are related via**

$$Z = \frac{g}{g_0} z \,. \qquad \textbf{(3)}$$

  - **If the local gravity is stronger than standard gravity, then $Z > z$.**
  - **If the local gravity is weaker than standard gravity, then $Z < z$.**
- **Gravity varies from around 9.79 to 9.82 m/s^2. Therefore, $g/g_0 \cong 1$, and for many applications we can ignore the difference between geopotential and geometric height, since $Z \cong z$.**
  - **But, keep in mind that they are different, and at times this difference, though small, is very important and cannot be neglected.**
- **Many equations, such as the hydrostatic equation or the hypsometric equation, can be written either in terms of geometric height or geopotential height. A convenient rule that applies most of the time is that**
  - **if the formula is written in geopotential height, Z, then g_0 is used for gravity.**
  - **if the formula is written in geometric height, z, then g is used for gravity.**
  - **Example: The hydrostatic equation can be written both ways, as shown below**

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial p}{\partial Z} = -\rho g_0$$

PRESSURE

- **Pressure is force per unit area.**
- **There are two types of pressure:**
  - ***Hydrostatic pressure*, which is just due to the weight of the air above you.**
  - ***Dynamic pressure*, which is due to the motion of the air.**
- **In meteorology, dynamic pressure is usually very small, and we will assume for now that atmospheric pressure is solely due to hydrostatic pressure.**
- **To find how pressure changes with height we start with the hydrostatic equation**

$$\frac{\partial p}{\partial z} \cong \frac{dp}{dz} = -\rho g ,$$

and the ideal gas law

$$p = \rho R_d T .$$

- **Eliminating density from these two equation gives**

$$\frac{1}{p}\frac{dp}{dz} = -\frac{g}{R_d T} . \qquad \textbf{(4)}$$

- **Integrating vertically from the surface to some height *z* we get**

$$\int_0^z \frac{1}{p}\frac{dp}{dz}\,dz = -\int_0^z \frac{g}{R_d T}\,dz$$

or

$$\int_{p_0}^{p(z)} \frac{dp}{p} = -\int_0^z \frac{g}{R_d T}\,dz$$

$$p(z) = p_0 \exp\left(-\frac{g}{R_d}\int_0^z \frac{1}{T}\,dz \right) . \quad \textbf{\textit{Pressure variation with height}} \qquad \textbf{(5)}$$

PRESSURE DECREASE IN AN ISOTHERMAL ATMOSPHERE

- **Absolute temperature varies by only 20% or so through the troposphere, so we can get an idea how pressure changes with height by assuming a constant temperature (isothermal atmosphere). If this is done, the expression for the pressure profile becomes**

$$p(z) = p_o \exp\left(-\frac{g}{R_d T} z \right) = p_0 \exp\left(-z/H_P \right) . \qquad \textbf{(6)}$$

- ***H_P*** **is the *pressure scale height* of the atmosphere, and is a measure of how rapidly the pressure drops with height. A larger scale height means a slower rate of decrease with height.**
  - **At *z* = *H_p* the pressure will have decreased to 37% of the surface value (*e^{-1}* = 0.368).**
  - **The pressure scale height is the *e-folding* scale for pressure.**

DENSITY PROFILE

- **We can also use the hydrostatic equation and the equation of state to find how density changes with height. We first start by differentiating the ideal gas law with respect to height to get**

$$\frac{dp}{dz} = R_d \left[\rho \frac{dT}{dz} + T \frac{d\rho}{dz} \right]. \quad \textbf{(7)}$$

From the hydrostatic equation we know that

$$\frac{dp}{dz} = -\rho g ,$$

and putting this into (7) we can write

$$R_d \left[\rho \frac{dT}{dz} + T \frac{d\rho}{dz} \right] = -\rho g . \quad \textbf{(8)}$$

Dividing (8) through by $R_d T \rho$ gives

$$\frac{1}{T}\frac{dT}{dz} + \frac{1}{\rho}\frac{d\rho}{dz} = -\frac{g}{R_d T}. \quad \textbf{(9)}$$

Integrating (9) from the surface to some level *z* we get

$$\rho(z) = \rho_0 \frac{T_0}{T(z)} \exp\left[-\frac{g}{R_d} \int_0^z \frac{1}{T} dz \right]. \quad \textbf{\textit{Density variation with height}} \ \textbf{(10)}$$

- ***Notice that the density and pressure profiles do not have the exact same functional dependence unless the atmosphere is isothermal*** **[$T(z) = T_0$], in which case**

$$\rho(z) = \rho_0 \exp\left(-z/H_P\right)$$
$$p(z) = p_0 \exp\left(-z/H_P\right).$$

AN IMPORTANT REITERATION!

- **Any of the equations we've derived above in terms of actual height *z* and actual gravity *g* can be converted to geopotential height *Z* by simply substituting standard gravity g_0 for actual gravity!**

THICKNESS AND THE HYPSOMETRIC EQUATION

- **In terms of geopotential height the hydrostatic equation is**

$$\frac{dp}{dZ} = -\rho g_0 . \qquad \textbf{(11)}$$

- **Substituting from the ideal gas law we have**

$$\frac{dp}{dZ} = -\frac{pg_0}{R_d T},$$

or

$$1 = -\frac{R_d}{g_0}\frac{T}{P}\frac{dp}{dZ} . \qquad \textbf{(12)}$$

- **Integrating (12) between two levels in the atmosphere gives**

$$\int_{Z_1}^{Z_2} dZ = -\frac{R_d}{g_0}\int_{Z_1}^{Z_2}\frac{T}{p}\frac{dp}{dZ}dZ$$

or

$$Z_2 - Z_1 = -\frac{R_d}{g_0}\int_{p_1}^{p_2} T\frac{dp}{p} . \qquad \textbf{(13)}$$

  - **Using the generalized mean value theorem of calculus (13)becomes**

$$Z_2 - Z_1 = -\frac{R_d}{g_0}\overline{T}\int_{p_1}^{p_2}\frac{dp}{p} \qquad \textbf{(14)}$$

where

$$\overline{T} = \frac{1}{\ln\left(p_1/p_2\right)}\int_{p_2}^{p_1} T\frac{dp}{p} . \qquad \textbf{(15)}$$

is the average temperature in the layer between p_1 and p_2.

- **So the formula for the geopotential distance between the two pressure levels is**

$$\Delta Z = Z_2 - Z_1 = \frac{R_d}{g_0}\overline{T}\ln\frac{p_1}{p_2} . \qquad \textbf{\textit{Hypsometric Equation}} \qquad \textbf{(16)}$$

- **The hypsometric equation tells us that the thickness between two pressure levels is directly proportional to the average temperature within the layer.**
- **We can use thickness as a measure of the average temperature of a layer.**
- **We can use contours of thickness in a similar manner to how we use isotherms.**
- **Colder layers are thinner, warmer layers are thicker.**

SEA-LEVEL PRESSURE REDUCTION

- **Weather observing stations measure *station pressure*, which must be converted to *sea-level pressure* for reporting and plotting on weather charts.**
- **The method of calculating the sea-level pressure is called *pressure reduction* or *reducing* the pressure to sea-level (confusing, because in most cases sea-level pressure is larger than station pressure).**
- **Sea-level pressure reduction is accomplished via the hysometric equation, (16), treating $Z_1 = 0$ as sea level, and $Z_2 = Z_{sta}$ as the geopotential height of the station.**
  - **This means $p_1 = p_{sl}$, the sea-level pressure**
  - **$p_2 = p_{sta}$, the station pressure**
- **Rearranging (16) with these definitions give**

$$p_{sl} = p_{sta} \exp\left(\frac{g_0 Z_{sta}}{R_d \bar{T}} \right) \qquad \textbf{(17)}$$

- **The differences in pressure reduction formulas used in various applications mainly lie in the assumptions regarding the layer-average temperature in the hypothetical atmospheric layer between the surface and sea level.**
- **For U. S. surface observing stations $\bar{T}$ is computed as follows:**
  - **The lapse rate between sea-level and the surface is a constant γ = 6.5 K/km.**
  - **The surface temperature, T_{sfc}, is the average of the current surface temperature and the 12-hour prior surface temperature,**

$$T_{sfc} = (T_{now} + T_{-12hr}) / 2 \,. \qquad \textbf{(18)}$$

  - **This means the layer-average temperature in the hypothetical layer between the station and the surface is**

$$\bar{T} = T_{sfc} + \frac{\gamma}{2} Z_{sta} \,, \qquad \textbf{(19)}$$

or combining (18) and (19),

$$\bar{T} = \frac{\left(T_{now} + T_{-12hr}\right)}{2} + \frac{\gamma}{2} Z_{sta} \,. \qquad \textbf{(20)}$$

- **In practice there are additional correction terms applied in the calculation of $\bar{T}$.**
  - **Humidity is accounted for by using the virtual temperature, T_v, instead of the actual temperature.**

- + **In practice, the Weather Service uses climatological humidity rather than observed humidity for this correction.**
- o **Empirical corrections are made for each specific station to account for:**
  - + **Variations in climatological lapse rates from summer to winter. This is called the 'plateau effect', because it is most extreme at high elevation stations.**
  - + **Local anomalies in lapse rate.**

ALTIMETER SETTING

- **Aircraft altimeters are essentially barometers that are calibrated to read altitude above mean sea level.**
- **By assuming a constant lapse rate and integrating (5) we obtain a relationship between pressure and height,**

$$p(z) = p_0 \left(\frac{T_0 - \gamma z}{T_0} \right)^{g/R_d\gamma}, \qquad \textbf{(21)}$$

which in terms of height vs pressure is

$$z = \frac{T_0}{\gamma} \left[1 - \left(\frac{p(z)}{p_0} \right)^{\gamma R_d / g} \right]. \qquad \textbf{(22)}$$

- **Equation (22) is what is used by altimeters to measure pressure and calculate altitude above sea level.**
- **However, we need to choose relevant values for both p_0 and T_0.**
- **We are really free to choose whatever values for p_0 and T_0 we want to use, and although the altitude wouldn't necessarily be correct, it would keep aircraft separated by altitude as long as everyone were using the same values for p_0 and T_0.**
- **What is done in practice is to choose values of p_0 and T_0 such that when $p(z)$ is equal to the station pressure p_{sta} then the altitude z will be equal to the station elevation, z_{sta},**

$$z_{sta} = \frac{T_0}{\gamma}\left[1 - \left(\frac{p_{sta}}{p_0}\right)^{\gamma R_d / g}\right]. \qquad \textbf{(23)}$$

- **With the proper altimeter setting, when the aircraft is on the airfield the altimeter will read the elevation of the airfield.**
- **The value of p_0 needed to achieve this is called the altimeter setting p_{alt}.**
- **The temperature profile is assumed to be the U.S. Standard Atmosphere, so that**

$$T(z) = T_0 - \gamma z \ , \qquad \textbf{(24)}$$

where γ is the standard lapse rate of 6.5K/km, and T_0 is the temperature at sea level.

- **We find T_0 as follows. We know that in the standard atmosphere**

$$\frac{dT}{dp} = \frac{dT}{dz}\frac{dz}{dp} = -\gamma\left(\frac{-1}{\rho g}\right) = \frac{\gamma R_d T}{pg}$$

or

$$\frac{dT}{T} = \frac{\gamma R_d}{g}\frac{dp}{p}. \qquad \textbf{(25)}$$

Integrating (25) from *standard sea-level pressure p = 1013.25 mb* and *standard sea-level temperature, T** = 288.15K, to pressure p_0 and T_0, yields**

$$T_0 = T^*\left(\frac{p_0}{p^*}\right)^{\gamma R_d / g} . \qquad \textbf{(26)}$$

- **Substituting (26) into (23) and then solving for p_0 (which is p_{alt}) results in**

$$p_{alt} = p_{sta}\left[1 + \frac{\gamma z_{sta}}{T^*}\left(\frac{p^*}{p_{sta}}\right)^{R_d\gamma/g}\right]^{g/R_d\gamma} . \qquad \textbf{(27)}$$

- ***NOTE! In practice, 0.3 mb is subtracted from the station pressure prior to using Eqn. (27). This accounts for the fact that when the aircraft is on the runway its altimeter is not sitting on the ground. The 0.3 mb correction comes from the assumption that the altimeter is located 3 meters above the ground.***

- ***Important Point*: The altitude from a pressure altimeter will always be just an approximation to true altitude. It should be most accurate for the altitude of the station elevation. For other altitudes it is approximate. The aim is to keep aircraft separated by altitude so that they don't collide. As long as all aircraft talking with the controller are using the same altimeter setting, then they can stay separated by altitude.**

SEA-LEVEL PRESSURE VS. ALTIMETER SETTING

- **Neither sea-level pressure nor altimeter setting are physical quantities which can be measured. They are defined and calculated quantities, and each is used for its own purpose.**
- **In a METAR observation, altimeter setting and sea-level pressure will often differ, because of their differing purposes, and assumptions made about the hypothetical atmospheric layer between the surface and sea level. In summary:**

Sea-level Pressure	*Altimeter Setting*
Used to create sea-level pressure charts for calculating horizontal pressure gradients.	**Used for calibrating aircraft altimeters so that they read the proper altitude.**
Uses 12-hour averaged station temperature.	**Doesn't use any information about station temperature.**
Assumes constant lapse rate of 6.5K/km in layer below surface.	**Assumes constant lapse rate of 6.5K/km in layer below surface.**
Corrects for humidity.	**No humidity correction.**
Corrects for 'plateau effect' at high elevations.	**No correction for plateau effect.**

- **Sometimes we would like to figure out the station pressure from a METAR observation.**
  - **We could do this by using either the sea-level pressure or the altimeter setting and solving the respective equations for station pressure.**
  - **Because the altimeter setting has fewer assumptions and doesn't use any current temperature or humidity information, calculating station pressure from altimeter setting should be a more reliable method.**
  - **Also, more METAR stations report altimeter setting than report sea-level pressure.**

- **The formula for calculating station pressure from altimeter setting is**

$$p_{sta} = \left(p_{alt}^{R_d\gamma/g} - \frac{\gamma z_{sta}}{T^*}\, p^{*R_d\gamma/g} \right)^{g/R_d\gamma} + 0.3 \text{ mb} \ . \qquad \textbf{(28)}$$

DISTRIBUTION OF MOLECULAR SPEEDS

- **In a sample of gas, not all molecules move at the same speed. Instead, there is a distribution of speeds.**
- **For an ideal gas, the speed distribution is given by the Maxwell-Boltzmann distribution function**

$$f(v) = 4\pi v^2 \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(-\frac{mv^2}{2kT} \right). \qquad \textbf{(29)}$$

- **The probability of finding a molecule with a speed between v_1 and v_2 is given by**

$$P(v_1; v_2) = \int_{v_1}^{v_2} f(v)\, dv. \qquad \textbf{(30)}$$

- **The most probable speed is the speed where the distribution function is a maximum.**
- **The mean speed is different than, but close to, the most probable speed.**

WHY IS THERE SO LITTLE HYDROGEN IN THE ATMOSPHERE?

- **An object cannot escape the gravitation pull of the Earth unless its speed exceeds the *escape velocity* (v_{esc} ~ 11,200 m/s).**
- **The probability of a molecule exceeding escape velocity is found by**

$$P_{escape} = \int_{v_{esc}}^{\infty} f(v) dv. \qquad \textbf{(31)}$$

- **For O_2 at 288 K, the probability of escape is virtually zero.**
- **For H_2 at 288 K the probability of escape is ~10^{-22}.**
  - **Though small, it is not inconsequential. Out of 1 mole of H_2, you can expect 60 or so molecules to achieve escape velocity.**
  - **Hydrogen constantly *leaks* into space**

IS THE UPPER ATMOSPHERE WELL MIXED?

- **The atmosphere is a mixture of several different gases. The most abundant are N_2, O_2, Ar, and CO_2.**
- **In order of molecular weight we have**

Molecule	Molecular Weight (g/mol)
CO_2	44
Ar	39
O_2	32
N_2	28

- **You would think that the atmosphere would stratify according to weight, with the heaviest molecules having the greatest concentration near the surface. Therefore, we would expect most of the CO_2 and Ar to be found near the surface.**
- **Without turbulence, molecular diffusion would dominate any vertical transport processes.**
  - **Molecular diffusion favors lighter molecules over heavier ones. Therefore, the lighter molecules would be better mixed through a layer than would the heavier molecules, which would remain near the bottom due to gravity.**
  - **Molecular diffusion is characterized by the *mean free path*, which is the average distance between collisions.**
    - **The shorter the mean free path, the less effective molecular diffusion becomes.**
    - **Mean free path increases as pressure (and density) decrease.**
- **If turbulence is present, mixing is accomplished very efficiently.**
  - **Turbulent mixing does not discriminate based on mass. All molecules are mixed just as effectively.**
  - **Turbulent mixing is characterized by the *mixing length*, which is the average length that an air parcel can travel and still retain its identity.**
- **If the mixing length is greater than the mean free path, turbulent mixing will dominate and all molecules will be well mixed.**

- **If the mean free path is greater than the mixing length, molecular diffusion will dominate and the heavier molecules will be found toward the bottom.**
- **Up to about 80 km or so, the mixing length is larger than the mean free path, so that turbulent mixing dominates and the atmosphere is well mixed.**
- **Above 80 km the mean free path becomes larger than the mixing length (because density is decreasing with altitude). Therefore, above 80 km molecular diffusion dominates and the atmosphere is no longer well mixed. Instead, it becomes stratifies with the heavier molecules concentrated at the bottom.**
- **The well-mixed region is called the *homosphere*.**
- **The stratified region is called the *heterosphere*.**
- **The transition layer between the two is called the *turbopause*.**

EXERCISES

1. **If the atmosphere was incompressible (density constant at all altitudes), 100 km thick, and had a surface pressure of 1000 mb, at what altitude would the pressure be 250 mb? Sketch the graph of pressure vs. altitude for this case and discuss how it compares with the real atmosphere.**

2. **Find an expression for *number density* (molecules per m^3) as a function of height for a general atmosphere, and for an isothermal atmosphere.**

3. **Explain why airplane cabins are pressurized.**

4. **If the thickness of the 1000 – 500 mb layer is 5400 gpm, what is the layer average temperature (in °C)?**