**15. Continuity and Compactness**

 (15.1)**Theorem**: Let $\left(X,d\_{1}\right), (Y,d\_{2})$ are metric spaces and $f:X\rightarrow Y$ is a continuous function. If $X $is a compact set , then $f(X)$ is a compact set in $Y$.

**Proof:** let $F=\{G\_{λ}\}\_{λ\in Λ}$ is an open cover of $f(X)$ in $Y$.

$$⟹f(X)⊆\bigcup\_{λ\in Λ}^{}G\_{λ}, G\_{λ}\in τ\_{λ}∀λ\in Λ$$

$$X⊆f^{-1}(f\left(X\right))⊆f^{-1}(\bigcup\_{λ\in Λ}^{}G\_{λ})=\bigcup\_{λ\in Λ}^{}f^{-1}(G\_{λ}) $$

Since $\bigcup\_{λ\in Λ}^{}f^{-1}(G\_{λ})⊆X⟹X=\bigcup\_{λ\in Λ}^{}f^{-1}(G\_{λ}) $

Since $f$ is a continuous $⟹f^{-1}(G\_{λ})$ is an open set in $ X ∀λ\in Λ$

$\{f^{-1}(G\_{λ})\}$ is an open cover of $X$

Since $X$ is a compact space $⟹∃ λ\_{1}, λ\_{2}, …,λ\_{n} \in Λ\ni X=\bigcup\_{i=1}^{n}f^{-1}(G\_{λ})$

$$⟹X=f^{-1}(\bigcup\_{i=1}^{n}G\_{λ})⟹f\left(X\right)=f(f^{-1}(\bigcup\_{i=1}^{n}G\_{λ}))⊆\bigcup\_{i=1}^{n}G\_{λ}$$

$⟹f\left(X\right)$ is a compact set in $Y$.

(15.2)**Corollary**: Let $\left(X,d\_{1}\right), (Y,d\_{2})$ are metric spaces and $f:X\rightarrow Y$ is a continuous function. If $A $is a compact set in $X$ , then $f(A)$ is a compact set in $Y$.

(15.3)**Example:** Let $\left(R,d\_{u}\right)$ is usual metric space and $f:R\rightarrow R$ is defined as $f\left(x\right)=2 ∀x\in R$.

We note that $f$ is a continuous, since $f$ is a constant and $A=\{1,2,3\}$ is a compact in $R$, since $A$ is a finite, but $f^{-1}\left(A\right)=R$ does not compact.

(15.4)**Theorem**: Let $\left(X,d\_{1}\right), (Y,d\_{2})$ are metric spaces $\ni X≅Y$, then $X$ is a compact space $⟺Y$ is a compact space.

**Proof:** since $X≅Y⟹∃ f:X\rightarrow Y$.

Let $X$ is a compact space, since $f$ is a continuous $⟹f\left(X\right)$ is a compact in $Y$.

Since$ f$ is a bijective $⟹f\left(X\right)=Y⟹Y$ is a compact space.

Now, let $Y$ is a compact space

Since $f^{-1}:Y\rightarrow X$ is a continuous $⟹f^{-1}\left(Y\right)=X$ is a compact.

(15.5)**Theorem**: Let $\left(X,d\right)$ is a compact space and $f:X\rightarrow R$ is a continuous function, then

1. $f$ is a bounded.
2. If $α=$ inf $\left\{f\left(x\right):x\in X\right\}, β=$ sup $\left\{f\left(x\right):x\in X\right\}$, then $∃ a,b\in X\ni f\left(a\right)=α, f\left(b\right)=β$.

**Proof:** (1) since $X$ is a compact space and $f$ is a continuous $⟹f(X)$ is a compact set in $R$.

Since every compact set in$ R$ is a closed and bounded $⟹f(X)$ is a bounded.

(2) since $f(X)$ is a bounded $⟹∃α, β\in R$ and since $f(X)$ is a closed $⟹α, β\in f(X)$

Put $a\in f^{-1}\left(\left\{α\right\}\right), b\in f^{-1}\left(\left\{β\right\}\right)⟹f\left(a\right)=α, f\left(b\right)=β$.

(15.6)**Theorem**: Let $\left(X,d\_{1}\right), (Y,d\_{2})$ are a metric spaces and $f:X\rightarrow Y$ is a continuous function. If $X$ is a compact space, then $f$ is an uniform continuous.

(15.7)**Corollary**: Let $\left(R,d\_{u}\right) $is usual metric space. If $f:[a,b]\rightarrow R$ is a continuous function, then $f$ is an uniform continuous.