**13. Uniform Continuity**

 (13.1)**Definition**: If $\left(X,d\_{1}\right),(Y,d\_{2})$ be metric spaces. We said that a function $f:X\rightarrow Y$ is an uniform continuous on $X$, if $∀ε>0 ∃δ >0\ni ∀x,y\in X$, then $d\_{1}(x,y)<δ⟹d\_{2}(f\left(x\right),f\left(y\right))<ε$.

(13.2)**Theorem**: Every uniform continuous is continuous.

**Proof:** let $\left(X,d\_{1}\right),(Y,d\_{2})$ are metric spaces and let a function $f:X\rightarrow Y$ is an uniform continuous. Let $x\_{0}\in X$, we must prove that $f$ be continuous at $x\_{0}$.

Let $ε>0, $since $f$ is an uniform continuous $⟹∃δ>0\ni ∀x,y\in X$, then

$d\_{1}(x,y)<δ⟹d\_{2}(f\left(x\right),f\left(y\right))<ε$, since $x\_{0}\in X⟹∀x\in X⟹d\_{1}(x,x\_{0})<δ⟹d\_{2}(f\left(x\right),f\left(x\_{0}\right))<ε⟹f$ is a continuous at $x\_{0}⟹f$ is a continuous.

(13.3)**Example:** Let $\left(R,d\_{u}\right)$ be usual metric space and a function $f:R\rightarrow R$ defined by $f\left(x\right)=x^{2}, x\in R$, then $f$ is continuous, but does not uniform continuous.

**Solution:** $ε>0\ni ∀δ>0 ∃x,y\in R$ and $\left|x-y\right|<δ⟹\left|f(x)-f(y)\right|>ε$

Let $δ>0, $(by Archimedes property) $∃k\in Z^{+}\ni \frac{1}{k}<δ$

Put $y=k+\frac{1}{k}, x=k⟹\left|x-y\right|=\frac{1}{k}<δ,$ but $\left|f(x)-f(y)\right|=2+\frac{1}{k^{2}}>2$

$⟹f$ does not uniform continuous.

**Real- Valued Functions**

(13.4)**Definition:** Let $f,g\in RV\left(X\right)=\left\{f:X\rightarrow R \right\}, λ\in R$. Define $f+g,λf,\frac{f}{g}, \left|f\right|$ as following:

* $\left(f+g\right)\left(x\right)=f\left(x\right)+g(x)$
* $\left(λf\right)\left(x\right)=λf(x)$
* $\left(fg\right)\left(x\right)=f\left(x\right)g(x)$
* $\left(\frac{f}{g}\right)\left(x\right)=\frac{f(x)}{g(x)}, g\left(x\right)\ne 0 ∀x\in X$
* $\left|f\right|\left(x\right)=\left|f(x)\right|$

(13.5)**Theorem:** If $f,g\in C(X)$ which denoted to set of all a continuous functions and defined from $(X,d)$ into $\left(R,d\_{u}\right)$ and $λ\in R$, then

1. $f+g\in C(X)$.
2. $λf\in C(X)$.
3. $fg\in C(X)$.
4. $\frac{f}{g}\in C(X)$.
5. $\left|f\right|\in C(X)$.

**Proof:** (1) let $x\_{0}\in X, ε>0$

Since $f,g\in C\left(X\right)⟹f:X\rightarrow R, g:X\rightarrow R$ are continuous functions

$⟹f,g$ are continuous at $x\_{0}$

Since $f:X\rightarrow R$ is continuous at $x\_{0}⟹∃δ\_{1}>0\ni ∀x\in X⟹d(x,x\_{0})<δ\_{1}⟹\left|f\left(x\right)-f(x\_{0})\right|<\frac{ε}{2}$

Since $g:X\rightarrow R$ is continuous at $x\_{0}⟹∃δ\_{2}>0\ni ∀x\in X⟹d(x,x\_{0})<δ\_{2}⟹\left|g\left(x\right)-g(x\_{0})\right|<\frac{ε}{2}$

Put $δ=$ min $\{δ\_{1},δ\_{2}\}⟹δ>0 ∀x\in X⟹d(x,x\_{0})<δ$

$$\left(f+g\right)\left(x\right)-\left(f+g\right)\left(x\_{0}\right)=\left(f\left(x\right)+g\left(x\right)\right)-(f\left(x\_{0}\right)+g\left(x\_{0}\right))$$

$$=\left( f\left(x\right)-f\left(x\_{0}\right)\right)+(g\left(x\right)-g\left(x\_{0}\right))$$

$$\left|\left(f+g\right)\left(x\right)-\left(f+g\right)\left(x\_{0}\right)\right|=\left| f\left(x\right)-f\left(x\_{0}\right)\right|+\left|(g\left(x\right)-g\left(x\_{0}\right))\right|<\frac{ε}{2}+\frac{ε}{2}=ε$$

$⟹f+g$ is continuous at $x\_{0}⟹f+g\in C(X)$.

**Boundedness**

(13.6)**Definition:** Let $(X,d)$ be metric space and $A⊆X$. We said that $A$ is bounded in $X$, if$ δ\left(A\right)=$ sup $\{d\left(x,y\right):x,y\in A\}<\infty $ or $B=\{d\left(x,y\right):x,y\in A\}$ is bounded in $R$. We say that $X$ is bounded space, if $δ\left(X\right)<\infty $.

(13.7)**Theorem:** Let $(X,d)$ be metric space and $A⊆X$. We said that $A$ is bounded in $X⟺∀x\_{0}\in A ∃k\in Z^{+}\ni d\left(x,x\_{0}\right)<k ∀x\in A$.

(13.8)**Example:** In usual metric space $\left(R,d\_{u}\right)$, we have

1. $A\_{1}=\left(a,b\right), A\_{2}=\left(a,b\right], A\_{3}=\left[a,b\right), A\_{4}=[a,b]$ be a bounded, since $δ\left(A\_{i}\right)=b-a ∀i=1,2,3,4$.
2. A space $R$ is unbounded, since $δ\left(R\right)=\infty $.

(13.9)**Definition:** Let $X,d)$ be metric space and $\left(R,d\_{u}\right)$ is usual metric space. We said that a function $f:X\rightarrow R$ is a bounded, if $∃ M\in R^{+}\ni \left|f\left(x\right)\right|\leq M ∀x\in X$.

**Intermediate Value Property**

(13.10)**Definition**: Let $\left(R,d\_{u}\right)$ is usual metric space. We said that $f:[a,b]\rightarrow R$ satisfies an intermediate value property, if $∀x,y\in \left[a,b\right], ∀s $between $f\left(x\right), f\left(y\right)∃ z$ between $x,y\ni f\left(z\right)=s$.

(13.11)**Example:** Let $\left(R,d\_{u}\right)$ be usual metric space and let a function $f:[a,b]\rightarrow R$ defined by $f\left(x\right)=x ∀ x\in [a,b]$, then a function $f$ satisfies an intermediate value property.

**Solution:** let $x,y\in [a,b]\ni x<y$ and let $f\left(x\right)<s<f(y)$

Since $f\left(x\right)=x ∀ x\in [a,b]⟹x<s<y$

Since $f\left(s\right)=s⟹f$ satisfies an intermediate value property.

(13.12) **Theorem** (**Intermediate Value Theorem**)

Let $\left(R,d\_{u}\right)$ is usual metric space. If a function $f:[a,b]\rightarrow R$ is a continuous, then $∀ s$ between $f\left(a\right), f\left(b\right), ∃ z$ in $\left[a,b\right] \ni f\left(z\right)=s$.

(13.13)**Example:** Let $\left(R,d\_{u}\right)$ be usual metric space. If a function $f:[0,1]\rightarrow R$ defined as $f\left(x\right)=\left\{\begin{array}{c}\sin(\frac{1}{x}), 0<x\leq 1\\0, x=0\end{array}\right.$, then $f$ satisfies an intermediate value property, but its discontinuous.