**9. Interior Points**

 (9.1) **Definition**: Let $(X,d)$ be a metric space, and $A⊆X$. We say that a point $x\in A$ is an interior point in $A$, if $∃$ an open set $G$ in $X \ni x\in G⊂A$. Or if $∃ r>0\ni B\_{r}\left(x\right)⊆A$.

(9.2) **Definition**: The set of all interior points in $A$ is denoted by

 $A^{∘}=\{x\in A:∃r>0,B\_{r}\left(x\right)⊆A\}$, so $A^{∘}⊂A$.

(9.3) **Notes**: From previous definitions, we deduce

1. $A^{∘}$ is an open set in $X$.
2. $A$ is an open set in $X$ iff $A^{∘}=A$.
3. $A^{∘}=(A^{∘})^{∘}$.

(9.4) **Example:** Let $(X,d)$ be an indiscrete metric space, and $A⊆X$. Calculate $A^{∘}$.

**Solution:** since $(X,d)$ be an indiscrete metric space $⟹d\left(x,y\right)=0 ∀x,y\in X$.

$$⟹A^{∘}=\left\{\begin{array}{c}∅, A\ne X\\X, A=X\end{array}\right.$$

(9.5) **Example:** Let $(X,d)$ be a discrete metric space, and $A⊆X$. Calculate $A^{∘}$.

**Solution:** since $(X,d)$ be a discrete metric space $⟹A$ is an open set in $X$.

$⟹A^{∘}=A$.

(9.6) **Example:** Let $(R,d\_{u})$ be usual metric space, and $A⊆R$, then we have

1. If $A=\left(a,b\right)⟹A^{∘}=\left(a,b\right)$.
2. If $A=(a,b]⟹A^{∘}=\left(a,b\right)$.
3. If $A=[a,b)⟹A^{∘}=\left(a,b\right)$.
4. If $A=[a,b]⟹A^{∘}=\left(a,b\right)$.
5. If $A=\left[a,b\right]∪[c,d]⟹A^{∘}=\left(a,b\right)∪(c,d)$.
6. If $A$ is a finite, then $A^{∘}=∅$.
7. If $A=N⟹A^{∘}=∅$.
8. If $A=Z⟹A^{∘}=∅$.
9. If $A=Q⟹A^{∘}=∅$.
10. If $A=\{\frac{1}{n},n\in N\}⟹A^{∘}=∅$.

(9.7) **Theorem:** Let $(X,d)$ be a metric space, and $A,B⊆X$, then we have

1. If $A⊆B⟹A^{∘}⊆B^{∘}$.
2. $(A∩B)^{∘}=A^{∘}∩B^{∘}$.
3. $A^{∘}∩B^{∘}⊂(A∪B)^{∘}$.

**Proof:** (1) let $x\in A^{∘}⟹∃r>0\ni B\_{r}\left(x\right)⊆A$.

Since $A⊆B⟹B\_{r}\left(x\right)⊆B⟹x\in B^{∘}⟹A^{∘}⊆B^{∘}$.

(9.8) **Note:** Not necessary that $(A∪B)^{∘}=A^{∘}∪B^{∘}$, for example

Let $(R,d\_{u})$ be usual metric space, and let $A=\left[0,1\right], B=[1,2]$, we have

$A^{∘}=\left(0,1\right), B^{∘}=\left(1,2\right)⟹1\notin A^{∘}, 1\notin B^{∘}⟹1\notin A^{∘}∪B^{∘}$, but $A∪B=[0,2]$

$⟹(A∪B)^{∘}=(0,2)$ and $1\in (A∪B)^{∘}⟹(A∪B)^{∘}\ne A^{∘}∪B^{∘}$.

(9.9) **Definition:** Let$ (X,d)$ be a metric space, and $A⊆X$. We said that $x\in X$ is a closure point of set $A$, if $∀r>0 ∃y\in A\ni d(x,y)<r$.

$\overbar{A}=\{x\in X:∀r>0, ∃y\in A\ni d(x,y)<r\}⟹A⊂\overbar{A}$.

(9.10) **Notes:** From previous definitions, we deduce

1. $\overbar{A}$ is a closed set in $X$.
2. $\overbar{A}$ is a closed set $⟺\overbar{A}=A$.
3. $\overbar{A}=̿$.

(9.11) **Example:** Let $(X,d)$ be indiscrete metric space, and $A⊆X$. Calculate $\overbar{A}$.

**Solution:** since$(X,d)$ be indiscrete metric space $⟹d\left(x,y\right)=0 ∀x,y\in X$

$$⟹\overbar{A}=\left\{\begin{array}{c}∅, A=∅\\X, A\ne ∅\end{array}\right.$$

(9.12) **Example:** Let $(X,d)$ be discrete metric space, and $A⊆X$. Calculate $\overbar{A}$.

**Solution:** since $(X,d)$ be discrete metric space $⟹A$ is a closed set in $X$.

$⟹\overbar{A}=A$.

(9.13) **Example:** Let $(R,d\_{u})$ be usual metric space, and $A⊆R$, then we have

1. If $A=\left(a,b\right)⟹\overbar{A}=[a,b]$.
2. If $A=(a,b]⟹\overbar{A}=[a,b]$.
3. If $A=[a,b)⟹\overbar{A}=[a,b]$.
4. If $A=[a,b]⟹\overbar{A}=[a,b]$.
5. If $A=N⟹\overbar{A}=N$.
6. If $A=Z⟹\overbar{A}=Z$.
7. If $A=Q⟹\overbar{A}=R$.
8. If $A=\left\{\frac{1}{n},n\in N\right\}⟹\overbar{A}=A∪\{0\}$.
9. If $A=\left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, …\right\}⟹\overbar{A}=A∪\{1\}$.

(9.14) **Theorem:** Let $(X,d)$ be a metric space, and $A,B⊆X$, then we have

1. If $A⊂B⟹\overbar{A}⊆\overbar{B}$.
2. $\overbar{A}∩\overbar{B}⊂\overbar{(A∩B)}$.
3. $\overbar{A}∩\overbar{B}⊂\overbar{(A∪B)}$.

**Proof:** (1) let $x\in \overbar{A}⟹∀r>0∃y\in A\ni d(x,y)<r$.

Since $A⊂B⟹y\in B⟹∀r>0 ∃y\in B\ni d(x,y)<r⟹x\in \overbar{B}⟹\overbar{A}⊆\overbar{B}$.

(9.15) **Note:** Not necessary that $\overbar{A}∩\overbar{B}=\overbar{(A∩B)}$, for example

Let $(R,d\_{u})$ be usual metric space, and let $A=(0,1), B=(1,2)$, we have

$\overbar{A}=\left[0,1\right], \overbar{B}=\left[1,2\right]⟹\overbar{A}∩\overbar{B}=\{1\}$, but $A∩B=∅⟹\overbar{\left(A∩B\right)}=∅$

$⟹\overbar{A}∩\overbar{B}\ne \overbar{(A∩B)}$.

(9.16) **Definition:** Let$ (X,d)$ be a metric space, and $A⊆X$. We said that $x\in X$ is a limit point of set $A$, if $∀r>0 ∃y\in A\ni y\ne x, d(x,y)<r$. The set of all limit points denoted by $A^{'}=\{x\in X:∀r>0, ∃y\in A\ni y\ne x,d(x,y)<r\}$.

(9.17) **Definition:** We said that $x\in X$ is an isolated point of $A$, if $x\in A, x\notin A^{'}$.

(9.18) **Definition:** We said that the set $A$ is an isolated set, if $A∩A^{'}=∅$.

(9.19) **Definition:** We said that the set $A$ is a perfect set, if $A=A^{'}$.

(9.20) **Definition:** We said that the set $A$ is a dense set, if $A⊂A^{'}$.

(9.21) **Theorem:** Let $(X,d)$ be a metric space, and $A⊆X$, then we have

1. If $A=∅⟹A^{'}=∅$.
2. If $x\in A^{'}⟹x\in (A\\{x\})^{'}$.
3. $A^{'}⊂\overbar{A}$.
4. $\overbar{A}=A∪A^{'}$.
5. If $A^{'}=∅⟹A$ is a closed in $X$.
6. $A$ is a closed in $X⟺A^{'}⊂A$.

**Proof:** (1) Since $∅∩(V\\{x\})=∅ ∀x\in X$ and $∀$ neighborhood $V$ of $X⟹∅^{'}=∅$.

(9.22) **Example:** Let $(X,d)$ be indiscrete metric space, and $A⊆X$. Calculate $A^{'}$.

**Solution:** (1) if $A=∅⟹A^{'}=∅$.

(2) if $A=X$, then we have

a. if $X$ contains one element $⟹A^{'}=∅$.

b. if $X$ contains more than one element $⟹A^{'}=X$.

3. if $A\ne ∅, A\ne X$.

a. if $A$ contains one element $⟹A=\left\{a\right\}⟹A^{'}=X\\left\{a\right\}$.

b. if $A$ contains more than one element $⟹A^{'}=X$.

(9.23) **Theorem:** Let $(X,d)$ be a metric space, and $A,B⊆X$, then we have

1. If $A⊆B⟹A^{'}⊆B^{'}$.
2. $(A∩B)^{'}⊂A^{'}∩B^{'}$.
3. $(A∪B)^{'}=A^{'}∪B^{'}$.$ $

**Proof:** (1) let $x\in A^{'}⟹∀r>0 ∃y\in A\ni y\ne x,d(x,y)<r$.

Since $A⊆B⟹y\in B⟹∀r>0 ∃y\in B\ni y\ne x,d\left(x,y\right)<r$

$⟹x\in B^{'}⟹A^{'}⊆B^{'}$.

(9.24) **Note:** Not necessary that $(A∩B)^{'}=A^{'}∩B^{'}$, for example

Let $X=\left\{a,b,c,d\right\}, (X,d)$ be indiscrete metric space, if $A=\left\{a\right\}, B=\{c,d\}$.

$$⟹A^{'}=\left\{b,c,d\right\}, B^{'}=X⟹A^{'}∩B^{'}=\left\{b,c,d\right\}⟹A∩B=∅⟹(A∩B)^{'}=∅$$

$⟹(A∩B)^{'}\ne A^{'}∩B^{'}$.

(9.25) **Definition:** Let$ (X,d)$ be a metric space, and $A⊆X$. We said that $x\in X$ is a boundary point of set $A$, if $∀r>0 ∃z\in A^{c}, y\in A\ni d\left(x,z\right)<r, d(x,y)<r$. The set of all boundary points denoted by

$∂(A)=\{x\in X:∀r>0, ∃y\in A, z\in A^{c}\ni ,d(x,y)<r,d\left(x,z\right)<r\}$.

(9.26) **Theorem:** Let$ (X,d)$ be a metric space, and $A⊆X$, then

1. $∂\left(A\right)=\overbar{A}∩(\overbar{A^{c}})⟹∂\left(A\right)⊂\overbar{A}$.
2. $∂\left(A\right)=∂\left(A^{c}\right)$.
3. $∂\left(A\right)$ is a closed set in $X$.
4. $A^{∘}=A\∂\left(A\right)$.
5. $\overbar{A}=A∪∂\left(A\right)$.

**Proof:** (1) let $x\in ∂\left(A\right)⟹∀$ open set $G$ in $X$ and $x\in G$

$$⟹G∩A\ne ∅, G∩A^{c}\ne ∅⟹x\in \overbar{A}, x\in \overbar{A^{c}}⟹x\in \overbar{A}∩(\overbar{A^{c}})$$

$$⟹∂\left(A\right)⊆\overbar{A}∩(\overbar{A^{c}})$$

By same way we prove that $\overbar{A}∩(\overbar{A^{c}})⊆∂\left(A\right)⟹∂\left(A\right)=\overbar{A}∩(\overbar{A^{c}})$.

(9.27) **Example:** Let $(X,d)$ be discrete metric space, and $A⊆X$. Calculate $∂\left(A\right)$.

**Solution:** since $A$ is a closed $⟹A=\overbar{A}$ and $A^{c}$ is closed $⟹\overbar{A^{c}}=A^{c}$

$⟹∂\left(A\right)=\overbar{A}∩\left(\overbar{A^{c}}\right)=A∩A^{c}=∅$.