**6. Metric Spaces**

 (6.1) **Definition**: If $X$ be a non-empty set. We said that $d:X×X\rightarrow R$ is a metric function on $X$, if

1. $d\left(x,y\right)\geq 0 ∀x,y\in X$.
2. $d\left(x,y\right)=0 ⟺x=y$.
3. $d\left(x,y\right)=d\left(y,x\right) ∀x,y\in X$.
4. $d\left(x,y\right)\leq d\left(x,z\right)+d\left(z,y\right) ∀x,y,z\in X$.

(6.2) **Note**: $(X,d)$ is called Metric Space.

(6.3) **Example**: Let $d\_{u}:R×R\rightarrow R$ is a function defined by $d\_{u}\left(x,y\right)=\left|x-y\right| ∀x,y\in R$, then $d\_{u}$ is a metric function, and $(R,d\_{u})$ is an usual metric space.

**Solution:** (1) let $x,y\in R⟹x-y\in R⟹\left|x-y\right|\geq 0⟹d\_{u}\left(x,y\right)\geq 0$.

(2) $d\_{u}\left(x,y\right)=0⟺\left|x-y\right|=0⟺x-y=0⟺x=y$.

(3) let $x,y\in R, d\_{u}\left(x,y\right)=\left|x-y\right|=\left|y-x\right|=d\_{u}\left(y,x\right)$.

(4) let $x,y,z\in R, x-y=\left(x-z\right)+(z-y)$

$$\left|x-y\right|=\left|\left(x-z\right)+(z-y)\right|\leq \left|x-z\right|+\left|z-y\right|$$

$d\_{u}\left(x,y\right)\leq d\_{u}\left(x,z\right)+d\_{u}\left(z,y\right)⟹$ $d\_{u}$ is a metric function on $R$.

(6.4) **Example**: Let $d:R×R\rightarrow R$ is a function defined by $d\left(x,y\right)=\left|x-y\right|+1 ∀x,y\in R$, does $d$ a metric function on$ R$?

**Solution:** let $x,y\in R\ni x=y⟹d\left(x,y\right)=1$ this means the second axiom does not satisfy $⟹d$ does not metric.

(6.5) **Example**: Let $X$ be a non-empty set and $d:X×X\rightarrow R$ is a function defined by $d\left(x,y\right)=\left\{\begin{array}{c}0, x=y\\1, x\ne y\end{array}\right.$ for all $x,y\in X$, then $d$ is a metric function and $(X,d)$ is called discrete metric space.

**Solution:**

(1) since $d\left(x,y\right)=0$ or $d\left(x,y\right)=1 ∀x,y\in X⟹d\left(x,y\right)\geq 0 ∀x,y\in X$.

(2) let $x,y\in X$, if $x=y⟹d\left(x,y\right)=0$ (by the definition of a function $d$), if $d\left(x,y\right)=0⟹x=y$, since if $x\ne y⟹d\left(x,y\right)=1$, but this is a contradiction $⟹ d\left(x,y\right)=0⟺x=y$.

(3) let $x,y\in X, d\left(x,y\right)=\left\{\begin{array}{c}0, x=y\\1, x\ne y\end{array}\right.=\left\{\begin{array}{c}0, y=x\\1, y\ne x\end{array}\right.=d(y,x)$.

(4) let $x,y,z\in X$

a. if $x=y⟹d\left(x,y\right)=0$, since $d\left(x,z\right)\geq 0, d\left(z,y\right)\geq 0⟹d\left(x,z\right)+ d\left(z,y\right)\geq 0⟹d\left(x,y\right)\leq d\left(x,z\right)+ d\left(z,y\right)$.

b. if $x\ne y⟹d\left(x,y\right)=1$, so $z\ne x$ or $z\ne y$, let $z\ne x⟹d\left(x,z\right)=1$ since $d\left(z,y\right)\geq 0⟹ d\left(x,z\right)+d\left(z,y\right)\geq 1⟹d\left(x,y\right)\leq d\left(x,z\right)+d\left(z,y\right)⟹d$ is a metric function on $X$.

(6.6) **Example**: Let $X$ be a non-empty set and $d:X×X\rightarrow R$ is a function defined by $d\left(x,y\right)=0$ for all $x,y\in X$, then $d$ is a metric function and $(X,d)$ is called indiscrete metric space.

(6.7)**Theorem:** Let $(X,d)$ be metric space, then

1. $\left|d\left(x,z\right)-d\left(z,y\right)\right|\leq d\left(x,y\right) ∀x,y,z\in X$.
2. $\left|d\left(x,y\right)-d\left(z,w\right)\right|\leq d\left(x,z\right)+ d\left(y,w\right)∀x,y,z\in X$.

**Proof:**

(1) $d\left(x,z\right)\leq d\left(x,y\right)+d\left(y,z\right)=d\left(x,y\right)+d\left(z,y\right)$

$d\left(x,z\right)-d\left(z,y\right)\leq d\left(x,y\right)…(1)$ Also

$$d\left(z,y\right)\leq d\left(z,x\right)+d\left(x,y\right)=d\left(x,z\right)+d\left(x,y\right)$$

$$d\left(z,y\right)-d\left(x,z\right)\leq d\left(x,y\right)$$

$$-d\left(x,z\right)-d\left(z,y\right)\leq d\left(x,y\right)$$

$d\left(x,z\right)-d\left(z,y\right)\geq -d\left(x,y\right)…(2)⟹$ from $\left(1\right), (2)⟹$

$-d(x,y)\leq d\left(x,z\right)-d\left(z,y\right)\leq d\left(x,y\right)⟹\left|d\left(x,z\right)-d\left(z,y\right)\right|\leq d\left(x,y\right)$.

(6.8)**Theorem:** Let $X$ be a non-empty set, then a function $d:X×X\rightarrow R$ be a metric function iff

1. $d\left(x,y\right)=0 $iff $x=y$.
2. $d\left(x,y\right)\leq d\left(x,z\right)+d\left(y,z\right) ∀x,y,z\in X$.

**Proof:** $⟹)$ suppose that $d$ is a metric function $⟹\left(1\right), (2)$ are satisfy (from a definition).

$⟸)$ if $\left(1\right), (2)$ are satisfy,

(1)Let $x,y\in X$ from $(2)$, we get

$d\left(x,x\right)\leq d\left(x,y\right)+d\left(x,y\right)=2d(x,y)$, but $d\left(x,y\right)=0$ from $(1)⟹d(x,y)\geq 0$.

(2) same the condition $(1)$.

(3) Let $x,y\in X$ from $(2)$, we get

 $d\left(y,x\right)\leq d\left(y,y\right)+d\left(x,y\right)=0+d\left(x,y\right)=d(x,y)$

$$d\left(x,y\right)\leq d\left(x,x\right)+d\left(y,x\right)=0+d\left(y,x\right)=d(y,x)⟹$$

$d\left(x,y\right)\leq d\left(y,x\right), d\left(y,x\right)=d\left(x,y\right)⟹d\left(x,y\right)=d(y,x)$.

(4) Let $x,y,z\in X$

$d\left(x,y\right)\leq d\left(x,y\right)+d\left(y,z\right)=d\left(x,z\right)+d(z,y)⟹d$ is a metric function on $X$.

(6.9)**Definition:** Let $X$ be metric space and $∅\ne A⊂X$. The diagonal of $A$ denoted by $δ(A)$ and defined by $δ\left(A\right)=$ sup $\{d\left(x,y\right):x,y\in A\}$, if $A=∅$ or $A$ contains on only one element, then $δ\left(A\right)=0$. The distance of point $p$ from $A$ denoted by $d(p,A)$ and defined by $d\left(p,A\right)=$ inf $\{d\left(p,x\right):x\in A\}$.

(6.10)**Note:** Its clear, if $p\in A$, then $d\left(p,A\right)=0$, and if $A=∅$, then $d\left(p,∅\right)=\infty $.

(6.11)**Definition:** Let $∅\ne B⊂X$. The distance between $A,B$ is denoted by $d(A,B)$ and defined by $d\left(A,B\right)=$ inf $\{d\left(x,y\right):x\in A, y\in B\}$. Its clear that, if$ A=∅$, then $d\left(∅,B\right)=\infty $.

(6.12)**Example:** Let $(R,d\_{u})$ be usual metric space and $A=\left[1,2\right), B=(2,4]$, we note that $δ\left(A\right)=1, δ\left(B\right)=2, d\left(\frac{5}{4},A\right)=0, d\left(\frac{3}{2},B\right)=\frac{1}{2}, d\left(\frac{9}{4},B\right)=0, d\left(5,B\right)=1, d\left(A,B\right)=0$.

(6.13)**Theorem:** Let $\left(X,d\right)$ be metric space and $∅\ne A⊂X$, then

1. $δ\geq 0, d\left(p,A\right)\geq 0 ∀p\in X, d(A,B)\geq 0$.
2. If $A$ is a finite, then $δ\left(A\right)=\infty $.
3. If $A∩B\ne ∅$, then $d\left(A,B\right)=0$.
4. $\left|d\left(p,A\right)-d\left(q,A\right)\right|\leq d\left(p,q\right) ∀p,q\in X$.
5. If $A⊂B$, then $d\left(p,A\right)\geq d\left(p,B\right) ∀p\in X$.

**Proof:**

(1) and (2) from the definition.

(3) since $A∩B\ne ∅⟹∃x\_{0}\in A∩B⟹x\_{0}\in A$ and $x\_{0}\in B$

$d\left(A,B\right)=$ inf $\left\{d\left(x,y\right):x\in A, y\in B\right\}\leq d\left(x,y\right) ∀ x\in A, ∀y\in B$

$⟹d(A,B)\leq d(x\_{0},x\_{0})$, but $d\left(A,B\right)\geq 0⟹d\left(A,B\right)=0$

(6.14)**Notes:**

1. If $d\left(p,A\right)=0$, then not necessary that $p\in A$.
2. If $d\left(A,B\right)=0$, then not necessary that $A∩B\ne ∅$.