1. **Outer Measure of a Bounded Set**

**Lengths of Bounded Open Sets**

(11.1) **Definition**: Let represents a family of all subsets of . We said that a function is outer on , if

1. .
2. .
3. If , then .
4. If sequence of subsets of , then .

(11.2) **Note:** .

(11.3) **Example:** let contains more than one element, and represents a family of all subsets of. Define a function as , then is outer measure on .

(11.4) **Definition**: Let represents a family of all bounded sets in . Define as . If is an open set in , then .

(11.5) **Theorem:**  is an outer measure on .

**Proof:** (1) let is a bounded set in .

Since is great lower bound of set its elements number non-negative, then .

(2) since is an open set, then .

(3) let are bounded sets in

Since

(4) let is a countable family of a bounded set in

an open set , and

is an open set and

(11.6) **Theorem**:

1. If is a set in then an open set contains and

and

1. If is a bounded set in and , then

**Proof:** (1) By definition of great lower bound an open set contains and

Since is an open set, then

Since

(11.7) **Theorem**:

1. If , then and in general, if is a countable set, then .
2. If is a neglected set, then .
3. If is a bounded and , then is a neglected set.
4. If , then .
5. If represents an irrational numbers in , then .

**Proof:** (1) , then contains and

From definition , we get .

**Inner Measure of a Bounded Set**

Let is a bounded set in , let represents a family of all closed bounded sets in and . a closed bound set, we note .

(11.8) **Definition**: Let a family of all bounded sets in . Define a function

as .

(11.9) **Theorem**: Let a family of all bounded sets in , then we have

1. .
2. .
3. If , then .
4. If is a sequence of disjoint sets in , then .
5. If is a closed set in , then .
6. If , then .
7. If is an open set in , then .

**Proof:** (1) let , since is a smallest upper bound of set its elements are non-negative, then .

**Measurable Sets**

(11.10) **Definition:** Let is an outer measure on . We said that is measurable respect to , if .

(11.11) **Note:**  is measurable .

(11.12) **Theorem:** Let is an outer measure on . If , then is measurable, then is measurable.

**Proof:** let , since , then

Also, , then

is measurable

Since , then is measurable.

(11.13) **Theorem:** Let is an outer measure on . If is measurable, then , are measurable.

**Proof:** let , since is measurable, then

is measurable, since is measurable,

is measurable.

(11.14) **Theorem:** Let is an outer measure on . If are measurable, then is measurable.

(11.15) **Corollary:** Let is an outer measure on . If are measurable, then are measurable.

**Proof:** (1) since are measurable, then are measurable

is measurable

is measurable.

(2) since is measurable.

(11.16) **Theorem:** Let is an outer measure on . If is measurable and , then .

**Proof:** since and is measurable.

Since , we get

(11.17) **Theorem**: Let is a bounded set in , then the following statements are equivalent.

1. is measurable.
2. an open bounded set in .
3. an open bounded set in contains .
4. .
5. an open set and closed set .

**Measure of Bounded Sets**

(11.18) **Definition**: Let represents a family of all subsets in . We say that a function is a measure on , if

1. .
2. If is a sequence of disjoint sets in , then .

(11.19) **Theorem**: Let represents a family of all measurable bounded set in . Define as , then measure on .

**Proof:** (1) let is measurable bounded set in , then .

(2) let is a sequence of measurable bounded set and disjoint, is bounded set in . Since is measurable , then measurable

(3) if , then

(4) if is a sequence subsets in , then .

(11.20) **Notes:**

* If is an open bounded set, then is measurable and .
* If is a neglected set, then is measurable and .
* If , then is measurable and .