The typical proof of the stability of the hyperbolic equation

Question: Investigate the stability of the hyperbolic equation $\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2$ which approximated by the following explicit scheme:

$$(\mathbf{u_{p,q+1}} - 2\mathbf{u_{p,q}} + \mathbf{u_{p,q-1}})/\mathbf{k^2} = (\mathbf{u_{p+1,q}} - 2\mathbf{u_{p,q}} + \mathbf{u_{p-1,q}})/\mathbf{h^2}$$

knowing that $\mathbf{E_{p,q} = e^{i\beta ph}\lambda^q}$.

Sol.

$$(u_{p,q+1} - 2u_{p,q} + u_{p,q-1})/k^2 = (u_{p+1,q} - 2u_{p,q} + u_{p-1,q})/h^2$$

We substitute the following error equation: $E_{p,q} = e^{i\beta ph}\lambda^q$ in the above equation:

$e^{i\beta ph}\lambda^{q+1} - 2e^{i\beta ph}\lambda^q + e^{i\beta ph}\lambda^{q-1} = r^2\{e^{i\beta ph}e^{i\beta h}\lambda^q - 2e^{i\beta ph}\lambda^q + e^{i\beta ph}e^{i\beta h}\lambda^q\}$, (eq.1)

where $r = k/h$, now division by $e^{i\beta ph}\lambda^q$

$$\lambda - 2 + \lambda^{-1} = -4\,r^2\ \sin^2\frac{\beta h}{2}$$

$$\lambda + \lambda^{-1} = 2 - 4\,r^2\ \sin^2\frac{\beta h}{2}$$

$$\lambda + \lambda^{-1} = 2(1 - 2\,r^2\ \sin^2\tfrac{\beta h}{2})$$

Let $A = 1 - 2r^2 sin^2\frac{\beta h}{2}$ …(eq. 2)

Hence $\lambda + \lambda^{-1} = 2A$

By multiply the two sides by , we get

$\lambda^2 + 1 = 2A\lambda$ or, $\lambda^2 - 2A\lambda + 1 = 0$,

to find the roots for the last equation:

$$\lambda = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2A) \mp \sqrt{(-2A)^2 - 4}}{2} = \frac{2A \mp \sqrt{4A^2 - 4}}{2}$$
$$= A \mp (A^2 - 1)^{1/2}$$

Hence the values of λ are:

$\lambda_1 = A + (A^2 - 1)^{\frac{1}{2}}$ and $\lambda_2 = A - (A^2 - 1)^{\frac{1}{2}}$

For stability $|\lambda| \le 1$

As r, k, β are real, $A \le 1$ by eq. 2

When $A < -1$, $|\lambda_2| > 1$, giving instability.

When $-1 \le A \le 1$, $A^2 \le 1$, $\lambda_1 = A + i(1 - A^2)^{\frac{1}{2}}$, $\lambda_2 = A - i(1 - A^2)^{\frac{1}{2}}$.

hence $|\lambda_1| = |\lambda_2| = \{A^2 + (1 - A^2)\}^{\frac{1}{2}} = 1$,

proving that equation (*) is stable for $-1 \le A \le 1$. By eq.(1), we then have:

$-1 \le 1 - 2r^2 sin^2\left(\frac{\beta h}{2}\right) \le 1$, The only useful inequality is:

$-1 \le 1 - 2r^2 sin^2\left(\frac{\beta h}{2}\right)$ giving $r \le 1$

Question: Investigate the stability of the fully-implicit finite difference equation: $\frac{(u_{p,q+1} - u_{p,q})}{k} = \frac{(u_{p-1,q+1} - 2u_{p,q+1} + u_{p+1,q+1})}{h^2}$ **approximating the parabolic equation** $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ **.**

Sol.

$$\frac{(u_{p,q+1} - u_{p,q})}{k} = \frac{(u_{p-1,q+1} - 2u_{p,q+1} + u_{p+1,q+1})}{h^2} \quad \ldots (*)$$

We use the error function $E_{p,q} = e^{i\beta ph}\lambda^q$

$$e^{i\beta ph}\lambda^{q+1} - e^{i\beta ph}\lambda^q = r\{e^{i\beta(p-1)h}\lambda^{q+1} - 2e^{i\beta ph}\lambda^{q+1} + e^{i\beta(p+1)h}\lambda^{q+1}\},$$

where r=k/h^2 . Division by $e^{i\beta ph}\lambda^q$ leads to:

$$\lambda - 1 = r\lambda(e^{-i\beta h} - 2 + e^{i\beta h})$$

$$\lambda - 1 = r\lambda(cos\beta h - 2 + e^{i\beta h})$$

$$\lambda - 1 = r\lambda(cos\beta h - i\ sin\beta h - 2 + cos\beta h + i\ sin\beta h)$$

$$\lambda - 1 = r\lambda(2\ cos\beta h - 2)$$

$$\lambda - 1 = r\lambda(2(1 - 2\ sin^2\left(\frac{\beta h}{2}\right) - 2)$$

$$\lambda - 1 = r\lambda(2 - 4\ sin^2\left(\frac{\beta h}{2}\right) - 2)$$

$$\lambda - 1 = -4r\lambda\ sin^2\left(\frac{\beta h}{2}\right)$$

Division by λ we get: $\lambda = \frac{1}{1 + 4r\ sin^2(\frac{\beta h}{2})}$

Since $|\lambda| \le 1$

From the last equation, we can see that the equation is stable for all positive r values according to the given condition ($|\lambda| \le 1$).