# Lecture (2)

# Checking the Numerical Stability

## 2.1 Introduction

The stability of numerical systems is closely related to numerical error. The finite difference scheme is stable if the errors made in a certain time step of the calculations do not cause an increasing growth in the errors as the calculations continue. A scheme with neutral stability is one in which errors are constant as the calculations go on. If the errors grow with time, then the numerical system is called unstable. For time-dependent problems, stability is ensured such that the numerical method produces a bounded solution when the solution of the differential equation is bounded. Stability, in general, can be difficult to check, especially when the equation considered is nonlinear.

The difference between a numerical solution and a true solution is

$$\boldsymbol{u_{i,j} - u(i\Delta x, j\Delta t)} \qquad \dots (2.1)$$

which represents the error of the numerical solution.

Because the exact solution $\boldsymbol{u_{i,j} \equiv u(x,t)}$ is not normally obtained we often cannot determine this error. However, we can estimate the accuracy of this scheme in terms of $\Delta x \; and \; \Delta t$.

**Definition**: The solution $u_{i,j}$ is stable as long as the error is bounded at increasing j.

There are three methods to determine the stability: (1) direct method (2) energy method. (3) Von Neumann's method

### 2.2 Direct Method

We know that the true solution is bounded and hence it is sufficient to test the numerical solution bound. We can write the advection equation which is

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} + c\frac{u_{i,j} - u_{i-1,j}}{\Delta x} = 0 \qquad \dots (2.2)$$

in this way

$$u_{i,j+1} = (1-\mu)u_{i,j} + \mu u_{i-1,j} \qquad \dots (2.3)$$

where $\mu = \frac{c\Delta t}{\Delta x}$ ... (2.4)

If $0 \leq \mu \leq 1$ which is considered as a necessary condition for convergence ($c\Delta t \leq \Delta x$) we have

$$|u_{i,j+1}| \leq (1-\mu)|u_{i,j}| + \mu|u_{i-1,j}| \qquad \ldots (2.5)$$

We will apply this relationship at the point i at the level j+1 where $|u_{i,j+1}|$ is maximum, i.e. $Max_{(i)}|u_{j+1}|$:

$$Max_{(i)}|u_{i,j+1}| \leq Max_{(i)}(1-\mu)|u_{i,j}| + Max_{(i)}\mu|u_{i-1,j}|$$

If we assume that $Max_{(i)}|u_{i,j-1}| = Max_{(i)}|u_{i,j}|$ then:

$$Max_{(i)}|u_{i,j+1}| \leq Max_{(i)}|u_{i,j}| - Max_{(i)}\mu|u_{i,j}| + Max_{(i)}\mu|u_{i,j}|$$

$$Max_{(i)}|u_{i,j+1}| \leq Max_{(i)}|u_{i,j}| \qquad \ldots (2.6)$$

This proves the boundedness of the numerical solution $u_{i,j}$ for all times and hence $0 \leq \mu \leq 1$ is a sufficient condition for the stability of equation (2.2) for this system. The stability condition became the same as the convergence condition. In other words, if the scheme is convergent then it is stable, and vice versa. Although the direct method is a simple, it is successful for only a limited number of systems.

## 2.3 Energy Method

If we know that the true solution is bounded, we will test whether $\sum_i (u_{i,j+1})^2$ is bounded also and hence all $u_{i,j}$ must be bounded and therefore one can proves the stability of the scheme. This method is called by energy because in physics $u^2$ is mostly proportional to some formula of energy (kinetic energy K.E.=1/2 $mv^2$).

By squaring equation (2.3) which is $(u_{i,j+1} = (1-\mu)u_{i,j} + \mu u_{i-1,j})$ and sum at i we get:

$$\sum_i (u_{i,j+1})^2 = \sum_i \left[(1-\mu)^2 (u_{i,j})^2 + 2\mu(1-\mu)u_{i,j}\, u_{i-1,j} + \mu^2 (u_{i-1,j})^2\right] \quad \ldots (2.7)$$

For simplicity, we consider u is periodic in x, and assume that the summation on a one complete cycle, then

$$\sum_i (u_{i-1,j})^2 = \sum_i (u_{i,j})^2 \ , \qquad \ldots (2.8)$$

Now we use Schwartz's inequality which states that:

$$\sum ab \leq \sqrt{\sum a^2}\sqrt{\sum b^2} \qquad \ldots (2.9)$$

then:

$$\sum_i u_{i,j}\, u_{i-1,j} \leq \sqrt{\sum_i (u_{i,j})^2}\,\sqrt{\sum_i (u_{i-1,j})^2} = \sum_i \left(u_{i,j}\right)^2. \qquad \dots (2.10)$$

By using (2.8) and (2.10) and if $1-\mu \geq 0$ then (2.7) will give the following inequality:

$$\sum_i (u_{i,j+1})^2 \leq [(1-\mu)^2 + 2\mu(1-\mu) + \mu^2] \sum_i \left(u_{i,j}\right)^2, \qquad \dots (2.11)$$

**or**

$$\sum_i (u_{i,j+1})^2 \leq \sum_i (u_{i,j})^2 \;. \qquad \dots (2.12)$$

Because $(1-\mu)^2 + 2\mu(1-\mu) + \mu^2$ is 1 (try it). Therefore, $1 \geq 1-\mu \geq 0$, together with the periodic boundary condition, is a sufficient condition for stability (2.3)

## 2.4 Von Neumann's Method

Sometimes it is called Fourier series method which is the most used one. However, it is not used in the nonlinear equations but we can use it the equations that which were converted to linearized from nonlinear equations. The solution to a linear equation can be expressed as the Fourier series. The Fourier series can be formulated in terms of sine and cosine but is algebraically easier to replace with their equivalent in complex exponential form. We will replace our common notations $u_{i,j}$ by $u(ph, qk) = u_{p,q}$ (in order not to confuse with complex number counter i) and that $h = \Delta x, k = \Delta t$.

$$A_n e^{in\pi x/l} = A_n e^{in\pi ph/Nh} = A_n e^{i\beta_n ph}, \quad \dots (2.13)$$

where $\beta_n = \frac{n\pi}{Nh} \quad and \quad Nh = l.$

To investigate the spread of error as t increases, we need to find a solution to the finite difference equation which is reduced to $e^{i\beta ph}$ when t=qk=0. We will assume that the error is:

$$E_{p,q} = e^{i\beta x} e^{\alpha t} = e^{i\beta ph} e^{\alpha qk} = e^{i\beta ph} \lambda^q \,, \qquad \dots (2.14)$$

where $\lambda = e^{\alpha k,} \quad and \quad \alpha,$ in general, is a complex constant

The upper expression can be reduced to $e^{i\beta ph}$ when q=0. The error will not increase by increasing of t provided that $|\lambda| \leq 1$ . Therefore our condition of stability will be $|\lambda| \leq 1$ in this method.

**Ex (1). Investigate the stability of the fully-implicit finite –difference equation,**

$$\frac{\left(u_{p,q+1}-u_{p,q}\right)}{k}=\frac{\left(u_{p-1,q+1}-2u_{p,q+1}+u_{p+1,q+1}\right)}{h^2} \qquad \dots (2.15)$$

approximating the parabolic equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ .

Solution: Since the error function $E_{p,q}$ satisfies the same difference equations as with $u_{p,q}$ , so substituting $E_{p,q}$ from equation (2.14) into equation (2.15) gives:

$$e^{i\beta ph}\lambda^{q+1}-e^{i\beta ph}\lambda^{q}=r\left\{e^{i\beta(p-1)h}\lambda^{q+1}-2e^{i\beta ph}\lambda^{q+1}+e^{i\beta(p+1)h}\lambda^{q+1}\right\},$$

where r=k/h² . Division by $e^{i\beta ph}\lambda^{q}$ leads to:

$$\lambda-1=r\lambda\left(e^{-i\beta h}-2+e^{i\beta h}\right)$$

$$=r\lambda(2cos\beta h-2)=-4r\lambda\, sin^2(\beta h/2)$$

Hence $\lambda=\frac{1}{1+4r\, sin^2(\frac{\beta h}{2})}$

From the last equation it is clear that the equation is stable for all r positive values according to the condition $|\lambda|\leq 1$ .

***Homework: write the above example in details.***

**Ex (2). The hyperbolic equation $\partial^2 u/\partial t^2=\partial^2 u/\partial x^2$ is approximated by the explicit scheme:**

$$(u_{p,q+1}-2u_{p,q}+u_{p,q-1})/k^2=(u_{p+1,q}-2u_{p,q}+u_{p-1,q})/h^2 \qquad \dots (*)$$

investigate its stability.

Solution: It is easy to show by the method in Ex.1 that the equation for λ is:

$\lambda^2-2A\lambda+1=0$ ,

where $A=1-2r^2 sin^2\left(\frac{\beta h}{2}\right)$, $r=k/h$ ... (**)

Hence the values of $\lambda$ are:

$\lambda_1=A+(A^2-1)^{\frac{1}{2}}$ and $\lambda_2=A-(A^2-1)^{\frac{1}{2}}$

For stability $|\lambda|\leq 1$

As r, k, β are real, $A\leq 1$ by eq. (**)

When $A<-1$, $|\lambda_2|>1$, giving instability.

When $-1\leq A\leq 1$, $A^2\leq 1$, $\lambda_1=A+i(1-A^2)^{\frac{1}{2}}$, $\lambda_2=A-i(1-A^2)^{\frac{1}{2}}$.

hence $|\lambda_1|=|\lambda_2|=\{A^2+(1-A^2)\}^{\frac{1}{2}}=1$ ,

proving that equation (*) is stable for $-1\leq A\leq 1$ . By eq.(**), we then have:

$-1\leq 1-2r^2 sin^2\left(\frac{\beta h}{2}\right)\leq 1$ , The only useful inequality is:

$$-1\leq 1-2r^2 sin^2\left(\frac{\beta h}{2}\right)$$

giving $r\leq 1$