Example 4.48 $C_{24} \cong C_{2^3.3} \cong C_{2^3} \times C_3 \ncong C_6 \times C_4 \cong C_2 \times C_3 \times C_4 \cong C_2 \times C_{2^2} \times C_3$.

Example 4.49

$$\begin{aligned} \mathbb{F}_7 C_{30} &\cong \mathbb{F}_7(C_2 \times C_3 \times C_5) \\ &\cong (\mathbb{F}_7 C_2)(C_3 \times C_5) \\ &\cong (\mathbb{F}_7 \oplus \mathbb{F}_7)(C_3 \times C_5) \\ &\cong (\mathbb{F}_7 \oplus \mathbb{F}_7)C_3)C_5) \\ &\cong (\mathbb{F}_7 C_3 \oplus \mathbb{F}_7 C_3)C_5) \\ &\cong (\mathbb{F}_7 C_3)C_5 \oplus (\mathbb{F}_7 C_3)C_5 \\ &\cong \ ? \end{aligned}$$

It is not obvious what $\mathbb{F}_7 C_3$ is ! (Lagrange's theorem doesn't help).

Hey Leo i thought I'd help you out here !!!

$\mathbb{F}_7 C_3 \cong \mathbb{F}_7 \oplus \mathbb{F}_7 \oplus \mathbb{F}_7$ (since $|\mathcal{U}(\mathbb{F}_7 C_3)| = 216 = 6^3$ and $\mathcal{U}(\mathbb{F}_7 C_3) \cong C_6 \times C_6 \times C_6$). So $\mathbb{F}_7 C_{30} \cong (\mathbb{F}_7 \oplus \mathbb{F}_7 \oplus \mathbb{F}_7)C_5 \oplus (\mathbb{F}_7 \oplus \mathbb{F}_7 \oplus \mathbb{F}_7)C_5 \cong \{\oplus_{i=1}^{3} \mathbb{F}_7\}C_5 \oplus \{\oplus_{i=1}^{3} \mathbb{F}_7\}C_5 \cong \{\oplus_{i=1}^{6} \mathbb{F}_7\}C_5 \cong \oplus_{i=1}^{6}\{\mathbb{F}_7 C_5\}$. Also $\mathbb{F}_7 C_5 \cong \mathbb{F}_7 \oplus \mathbb{F}_{7^4}$ (since $|\mathcal{U}(\mathbb{F}_7 C_5)| = 14400 = (7-1)(7^4-1)$ and $\mathcal{U}(\mathbb{F}_7 C_5) \cong C_6 \times C_{2400}$) so $\mathbb{F}_7 C_{30} \cong \oplus_{i=1}^{6}\{\mathbb{F}_7 \oplus \mathbb{F}_{7^4}\}$.

$$\therefore \mathbb{F}_7 C_{30} \cong \oplus_{i=1}^{6} \mathbb{F}_7 \oplus_{i=1}^{6} \mathbb{F}_{7^4}$$

Example 4.50 $\mathbb{F}_5 D_{12} \cong \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus M_2(\mathbb{F}_5) \oplus M_2(\mathbb{F}_5)$ *or* $\mathbb{F}_5 D_{12} \cong \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus \mathbb{F}_5 \oplus M_2(\mathbb{F}_{5^2})$.

We mentioned before that $D_{12} \cong D_6 \times C_2$. $\therefore \mathbb{F}_5 D_{12} \cong \mathbb{F}_5(C_2 \times D_6) \cong (\mathbb{F}_5 C_2)D_6 \cong (\mathbb{F}_5 \oplus \mathbb{F}_5)D_6 \cong \mathbb{F}_5 D_6 \oplus \mathbb{F}_5 D_6$.

$$\therefore \mathbb{F}_5 D_{12} \cong (\mathbb{F}_5 \oplus \mathbb{F}_5 \oplus M_2(\mathbb{F}_5)) \oplus (\mathbb{F}_5 \oplus \mathbb{F}_5 \oplus M_2(\mathbb{F}_5)) \cong \oplus_{i=1}^{4} \mathbb{F}_5 \oplus \oplus_{j=1}^{2} M_2(\mathbb{F}_5).$$

Note : $\mathbb{C}S_3 \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ but $\mathbb{Q}S_3 \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{H}$ where $\mathbb{H}$ is the division ring of quaternions over $\mathbb{Q}$.

The End