Proof. Clearly $RG \cong R(G/G') \oplus \Delta(G,G')$. Now it is also clear that $R(G/G') \cong \oplus$ sum of the commutative summands of RG. It suffices to show that $\Delta(G,G')$ contains no commutative summands.

Assume $\Delta(G,G') \cong A \oplus B$ where A is commutative (and $\neq \{0\}$). Thus $RG \cong R(G/G') \oplus A \oplus B$. Now $RG/B \cong R(G/G') \oplus A$ (check). (In general, $R \cong C \oplus D \Longrightarrow R/C \cong D$). So RG/B is commutative, so by the previous lemma , $\Delta(G,G') \subset B$. Thus $\Delta(G,G') \cong A \oplus B \subset B$ which is a cotradiction. ∎

Definition 4.32 $D_{2n} =< x, y \,|\, x^n = y^2 = 1, yxy = x^{-1} >$ *is called the* ***dihedral group*** *of order* $2n$.

Note : $D_{2.3} = D_6 \cong S_3$.

Example 4.33 $\mathbb{F}_3 D_{10}$. *Note that Maschke applies so* $\mathbb{F}_3 D_{10} \cong \oplus_{i=1}^{s} M_{n_i}(D_i)$ $\cong \oplus_{i=1}^{s} M_{n_i}(K_i)$ *(where* K_i *are finite fields containing* $\mathbb{F}_3$*)* $\mathbb{F}_3 \oplus \oplus_{i=1}^{t} M_{n_i}(K_i)$

Note : $D_{10} =< x, y \,|\, x^5 = y^2 = 1, yxy = x^4 >$, $\therefore [x,y] = x^{-1}y^{-1}xy = x^4yxy = x^4.x^4 = x^8 = x^3$. $\therefore D_{10}{}' >< x^3 >$ *so* $D_{10}{}' >< x > \cong C_5$.

$\therefore \mathbb{F}_3 D_{10} \cong \mathbb{F}_3(D_{10}/D_{10}{}') \oplus$ *non-commutative piece* $\cong \mathbb{F}_3 C_2 \oplus$ *non-commutative piece* $\cong \mathbb{F}_3 \oplus \mathbb{F}_3 \oplus$ *non-commutative piece. By counting dimensions we get either*

$$\mathbb{F}_3 D_{10} \cong \mathbb{F}_3 \oplus \mathbb{F}_3 \oplus M_2(\mathbb{F}_3) \oplus M_2(\mathbb{F}_3)$$

or

$$\mathbb{F}_3 D_{10} \cong \mathbb{F}_3 \oplus \mathbb{F}_3 \oplus M_2(\mathbb{F}_{3^2})$$

Example 4.34 $\mathbb{F}_5 D_{12}$. $5 \nmid 12$ *so maschke applies.* $\mathbb{F}_5 D_{12} \cong \oplus_{i=1}^{s} M_{n_i}(D_i) \cong$