Proof. Let $H = < s >$. Let $1 \neq h \in H \therefore h = {s_1}^{\varepsilon_1} {s_2}^{\varepsilon_2} \dots {s_r}^{\varepsilon_r}$, where $s_i \in S$ and $\varepsilon_i = \pm 1$. Recall

$$\Delta_R(G, H) = \{\sum_{h \in H} \alpha_h(h-1) \,|\, \alpha_h \in RG\}.$$

So we must show that $h \in H \Longrightarrow h - 1 \in RG\{s-1 \,|\, s \in S\}$. Now $h - 1 = {s_1}^{\varepsilon_1} \dots {s_r}^{\varepsilon_r} - 1 = ({s_1}^{\varepsilon_1} \dots {s_{r-1}}^{\varepsilon_{r-1}})({s_r}^{\varepsilon_r} - 1) + ({s_1}^{\varepsilon_1} \dots {s_{r-1}}^{\varepsilon_{r-1}} - 1)$.

If $\varepsilon_r = 1$ then we are done (by induction on r). If $\varepsilon_r = -1$, then use $s_r^{-1} - 1 = s_r^{-1}(1 - s_r) = -s_r^{-1}(s_r - 1)$ and $h - 1 \in RG\{s - 1 \,|\, s \in S\}$.

Note : we used $x^{-1} - 1 - x^{-1}(1 - x)$ and $xy - 1 = x(y - 1) + (x - 1)$ and induction on r. ■

Recall : If $N \lhd G$ then G/N is commutative if and only if $G' < N$.

Lemma 4.30 *Let R be a commutative ring and I an ideal of RG. Then RG/I is commutative if and only if $\Delta(G, G') \subset I$.*

Proof. Let $I \lhd RG$, R commutative. ($\Rightarrow$). RG/I commutative $\Longrightarrow \forall\, g, h \in G$ we have $gh - hg \in I$. $gh = hg = hg(g^{-1}h^{-1}gh - 1) = hg([h, g] - 1) \in I$. $\Longrightarrow [h, g] - 1 \in I$. $\therefore \Delta(G, G') \subset I$ (by the previous lemma).

($\Leftarrow$). Assume $\Delta(G, G') \subset I$. Then $gh - hg = hg([h, g] - 1) \in \Delta(G, G') \subset I$. $\therefore gh = hg \bmod \Delta(G, G')$, so g and h commute modulo I so RG/I is commutative. ■

Proposition 4.31 *Let G be finite. Let RG be semisimple (i.e. $RG \cong \oplus_{i=1}^{s} M_{n_i}(D_i)$). Let $e_{G'} = \frac{1}{|G'|}.\widehat{G'}$. Then*

$$RG \cong RGe_{G'} \oplus RG(1 - e_{G'}) \cong R(G/G') \oplus \Delta(G, G').$$

Here $R(G/G')$ is the direct sum of all the commutative summands of the decomposition of RG and $\Delta(G, G')$ is the direct sum of all the non-commutative summands of the decomposition of RG.