Lemma 4.26 *Let $H < G$ and R a ring. Then $\mathcal{A}nn_r(\Delta(G,H)) \neq 0$ iff H is finite. In this case*

$$\mathcal{A}nn_l(\Delta(G,H)) = \widehat{H}.RG.$$

Furthermore, if $H \lhd G$ then $\widehat{H}$ is central in RG and

$$\mathcal{A}nn_r(\Delta(G,H)) = \mathcal{A}nn_l(\Delta(G,H)) = \widehat{H}.RG = RG.\widehat{H}$$

Proof. ($\Rightarrow$). Let's assume that $\mathcal{A}nn_r(\Delta(G,H)) \neq 0$ and let $0 \neq \alpha = \sum a_g g \in \mathcal{A}nn_r(\Delta(G,H))$. So if $h \in H$ we get $(h-1)\alpha = 0$ (since $h - 1 \in \Delta(G,H)$).

$\Longrightarrow h\alpha = \alpha$, so $\sum a_g g = \sum a_g h_g$. Let $g_0 \in \operatorname{supp}\alpha$, so $\alpha_{g_0} \neq 0$. So $hg_0 \in \operatorname{supp}\alpha \ \forall\, h \in H$. But $\operatorname{supp}\alpha$ is finite so H is finite.

($\Leftarrow$). Conversely, let H be finite. $\therefore \widehat{H}$ exists and $\widehat{H} \in \mathcal{A}nn_r(\Delta(G,H))$. $\therefore \mathcal{A}nn_r(\Delta(G,H)) \neq 0$.

" In this case ... " : Assume that $\mathcal{A}nn_r(\Delta(G,H)) \neq 0$ i.e. H is finite. Let $0 \neq \alpha = \sum a_g g \in \mathcal{A}nn_r(\Delta(G,H))$. As before $\alpha_{g_0} = \alpha_{hg_0}$.

Now we can partition G into it's cosets (generated by H) to get

$$\begin{aligned}
\alpha &= \sum a_g g \\
&= a_{g_0}\widehat{H}g_0 + a_{g_1}\widehat{H}g_1 + \cdots + a_{g_t}\widehat{H}g_t \\
&= \widehat{H}\left(\sum_{i=1}^{t} a_{g_i}g_i\right) \\
&= \widehat{H}B \ \exists\, B \in RG \\
\therefore &\quad \mathcal{A}nn_r(\Delta(G,H)) \subset \widehat{H}.RG.
\end{aligned}$$

Clearly $\widehat{H}.RG \subset \mathcal{A}nn_r(\Delta(G,H))$ (since $(h-1)\widehat{H}RG = 0.RG = 0$).
"Furthermore ..." easy. ■

Proposition 4.27 *Let R be a ring and $H \lhd G$. If $|H|$ is invertible in R then letting $e_H = \dfrac{1}{|H|}.\widehat{H}$ we have*

$$RG \cong RG.e_H \oplus RG(1 - e_H)$$

where $RG.e_H \cong R(G/H)$ and $RG(1-e_H) \cong \Delta(G,H)$.