(ii) $e_i = (0, \ldots, 0, 1.e_i, 0, \ldots, 0) \in L_1 \oplus \cdots \oplus L_t$. $\therefore e_ie_j =$
$(0, \ldots, 0, 1.e_i, 0, \ldots, 0)(0, \ldots, 0, 1.e_j, 0, \ldots, 0) = (0, \ldots, 0) = 0.$

(iv) Assume that (iv) does not hold, so $e_i = e_i' + e_i''$, (where e_i' and e_i'' are idempotents such that $e_i'e_i'' = 0 = e_i''e_i'$). Note that $R = \oplus_{i=1}^{t} L_i = \oplus_{i=1}^{t} Re_i$. $Re_i \subset L_i$ since $e_i \in L_i$ and L_i is a left ideal. Show $L_i \subset Re_i$. Let $a \in L_i$. Then $a = a.1 = a(e_1 + e_2 + \cdots + e_t) = ae_1 + ae_2 + \cdots + ae_t$.

$\Longrightarrow \underbrace{a - ae_i}_{\in L_i} = \underbrace{ae_1 + ae_2 + \cdots + ae_{i-1} + ae_{i+1} + \cdots + ae_t}_{L_1 \oplus L_2 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_t}$.

$\therefore a - ae_i = 0 \Longrightarrow a = ae_i \in Re_i$ and so $Re_i = l_i$.
$L_i = Re_i = R(e_i' + e_i'') = Re_i' \oplus Re_i''$. Now Re_i' and Re_i'' are left ideal so L_i is not minimal. This is a contradiction.

($\Leftarrow$) skip. ∎

Note : A set of idempotents $\{e_1, e_2, \ldots, e_t\}$ with properties (i),(ii) and (iii) above are called **complete family of orthogonal idempotents**. If $\{e_1, e_2, \ldots, e_t\}$ has the property of (i)-(iv), then it is called a set of **primitive idempotents**.

Theorem 4.2 (***Wedderburn-Artin Theorem***) *R is a semisimple ring if and only if R can be decomposed as a direct sum of finitely many matrix rings over division rings.*

$$i.e.\ R \cong M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus \cdots \oplus M_{n_s}(D_s)$$

where D_i is a division ring and $M_{n_i}(D_i)$ is the ring of $n_i \times n_i$ matrices over D_i.

Theorem 4.3 *Let R be a semisimple ring. Then the wedderburn-artin decomposition above is unique.*

$$i.e.\ R \cong \oplus_{i=1}^{s} M_{n_i}(D_i) \cong \oplus_{i=1}^{t} M_{m_i}(D_{i'}) \Longrightarrow s = t$$

and after permuting indices $n_i = m_i$ and $D_i = D_{i'}$ $\forall\, i \in 1, \ldots, s$.

Theorem 4.4 (***Maschke's Theorem***) *Let G be a group and R a ring. Then RG is semisimple if the following conditions hold :*