Chapter 4

Decomposition of RG

Theorem 4.1 *Let R be a semisimple ring with*

$$R = \oplus_{i=1}^{t} L_i$$

where the L_i are minimal left ideals. Then $\exists\, e_1, e_2, \ldots, e_n \in R$ such that

(*i*) *$e \neq 0$ is an idempotent for $i = 1, \ldots, t$.*

(*ii*) *If $i \neq j$, then $e_i e_j = 0$.*

(*iii*) *$e_1 + e_2 + \cdots + e_t = 1$.*

(*iv*) *e_i cannot be written as $e_i = e_i' + e_i''$ (where e_i' and e_i'' are idempotents such that $e_i' e_i'' = 0 = e_i'' e_i'$).*

Conversely, if $\exists\, e_1, e_2, \ldots, e_t \in R$ satisfying the four conditions above, then the left ideals $L_i = Re_i$ are minimal and $R = \oplus_{i=1}^{t} L_i$ (and $\therefore R$ is semisimple).
Proof. ($\Rightarrow$). Let $R = \oplus_{i=1}^{t} L_i$, where L_i is a minimal left ideal (for $i = \{1, 2, \ldots, t\}$).

(iii) $1 \in R$, so $1 = e_1 + e_2 + \cdots + e_t \; \exists\, e_i \in L_i$.

(i) Indeed, $e_i = 1.e_i = (e_1 + e_2 + \cdots + e_t)e_i = e_1 e_i + e_2 e_i + \cdots + {e_i}^2 + \cdots + e_t$.
$\Longrightarrow \underbrace{e_i - {e_i}^2}_{\in L_i} = \underbrace{e_1 e_i + e_2 e_i + \cdots + e_{i-1} e_i + e_{i+1} e_i + \cdots + e_t}_{L_1 \oplus L_2 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_t}$.
$\therefore e_i - {e_i}^2 \in L_1 \oplus L_2 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_t \Longrightarrow e_i - {e_i}^2 = 0 \Longrightarrow e_i = {e_i}^2$.