So the matrix of $\mathcal{T}(g)$ has all zero's in it's main diagonal. Hence the $\mathrm{tr}(\mathcal{T}(g)) = 0 \; \forall \; g \in G$ except for $g = 1$.

$$
\begin{aligned}
\therefore \mathrm{tr}\,(\mathcal{T}(\gamma)) &= \mathrm{tr}\left(\sum_{i=1}^{n} c_{g_i} g_i\right) \\
&= \sum_{i=1}^{n} c_{g_i} \mathrm{tr}\,(\mathcal{T}(g_i)) \\
&= c_{g_1} \mathrm{tr}\,(\mathcal{T}(g_1)) + c_{g_2} \mathrm{tr}\,(\mathcal{T}(g_2)) + \cdots + c_{g_n} \mathrm{tr}\,(\mathcal{T}(g_n)) \\
&= c_{g_1} \mathrm{tr} \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} + 0 + \cdots + 0 \\
&= c_{g_1}.|G| \\
&= c_1.|G|
\end{aligned}
$$

■

Theorem 3.10 (***Berman-Higman***) *Let* $\gamma = \sum_{g \in G} c_g g$ *be a unit of finite order in* $\mathbb{Z}G$*, where* G *is a finite group and* $c_1 \neq 0$*. Then* $\gamma = \pm 1 = c_1$*.*

Proof. Let $|G| = n$ and let $\gamma^m = 1$. Considering $\mathbb{Z}G$ as a subring of $\mathbb{C}G$, we will consider it's left regular representation and apply the previous lemma. Then $\mathrm{tr}\,(\mathcal{T}(\gamma)) = n.c_1$. Now $\gamma^m = 1$ therefore all the eigenvalues of $\mathcal{T}(\gamma)$ are the n^{th} roots of unity.

$$\therefore \mathrm{tr}\,(\mathcal{T}(\gamma)) = \mathrm{tr}\,(\mathcal{T}\left(\sum_{i=1}^{n} c_{g_i} g_i\right)) = \sum c_g \mathrm{tr}\,(\mathcal{T}(g)) = \sum (\text{eigenvalue of } \mathrm{tr}\,(\mathcal{T}(\gamma)))$$

Now $\mathcal{T}(\gamma)$ is similar to a diagonal matrix D ($\mathcal{T}(\gamma) \backsim D$). So $\mathrm{tr}\,(\mathcal{T}(\gamma)) = \mathrm{tr}\, D$ $= \sum$ diagonal elements of $D = \sum$ eigenvalues of $D = \sum$ eigenvalue of $\mathcal{T}(\gamma)$