We now extend $\mathcal{T}$ to a group ring representation. $\mathcal{T} : RG \longrightarrow M_n(R)$ where

$$\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \mathcal{T}(g) = \sum_{g\in G}(a_g I_{n\times n}) = (\sum_{g\in G} a_g) I_{n\times n} = \varepsilon\left(\sum_{g\in G} a_g g\right) I_{n\times n}$$

Example 3.7 *Let* $2g + (-2h) \in RG$. *Then* $\mathcal{T}(2g + (-2h))$

$$= \varepsilon(2g+(-2h))I_{n\times n} = (2+-2)I_{n\times n} = 0I_{n\times n} = 0_{n\times n} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Example 3.8 *Let* $2g + (-2h) + 21 \in RG$. *Then* $\mathcal{T}(2g + (-2h) = 21)$

$$= \varepsilon(2g+(-2h)+21)I_{n\times n} = (2+-2+21)I_{n\times n} = 21I_{n\times n} = \begin{pmatrix} 21 & 0 & 0 & \dots & 0 \\ 0 & 21 & 0 & \dots & 0 \\ 0 & 0 & 21 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 21 \end{pmatrix}.$$

Note $\mathcal{T} : RG \longrightarrow M_n(R)$ is onto and the $Ker(\mathcal{T}) = \Delta(RG)$.

Lemma 3.9 *Let G be a finite group and K a field. Let $\mathcal{T}$ be the left regular representation of KG and let $\gamma = \sum_{g\in G} c_g g \in KG$. Then the trace of $\mathcal{T}(\gamma)$ is*

$$tr\,(\,\mathcal{T}(\gamma)\,) = |G|.c_1$$

(where c_1 is the coefficient of $g_1 = 1$. For example if $\gamma = 2 + 3g + 4h \in KG$, then $c_1 = 2$).

Proof. The traces of similar matrices are the same and so $\mathrm{tr}\,(\,\mathcal{T}(\gamma)\,)$ is independant of choice of basis. Fix the basis $G = \{g_1 = 1, g_2, \dots, g_n\}$ (a K-basis of KG). $\therefore \mathcal{T}(\gamma) = \mathcal{T}\left(\sum_{g\in G} c_g g\right) = \sum_{g\in G} c_g \mathcal{T}(g) = \sum_{i=1}^{n} c_{g_i} \mathcal{T}(g_i)$. If $g \neq 1$, then $gg_i \neq g_i\ \forall\ i$ so g permutes the basis of KG.