Lemma 3.5 *Let K be a field and G a finite group.*

(*i*) *If $\alpha \in KG$ is nilpotent (i.e. $\exists m \in N$ such that $\alpha^m = 0$), then the eigenvalues of $(\mathcal{T}(\alpha))$ are all zero.*

(*ii*) *If $\beta \in KG$ is a unit of finite order (i.e. $\exists n \in N$ such that $\beta^n = 1$), then the eigenvalues of $(\mathcal{T}(\alpha))$ are all n^{th} roots of unity.*

(*iii*) *If $f(\gamma) = 0$, $\exists \gamma \in KG$ and $\exists f \in K[x]$ (the set of all polynomials over K) then $f(\lambda_i) = 0 \; \forall$ eigenvalues λ_i of $(\mathcal{T}(\gamma))$*

Proof. Note that $(iii) \Longrightarrow (i)$ and (ii). **(i)** Let $\alpha \in KG$ with $\alpha^m = 0$. Let λ be an eigenvalue of $(\mathcal{T}(\alpha))$ i.e. $(\mathcal{T}(\alpha))X = \lambda X$ where X is a $n \times 1$ column vector with entries in K. Now $(\mathcal{T}(\alpha))^m.X = \lambda^m.X$. $(\mathcal{T}(\alpha))^m.X = \mathcal{T}(\alpha)^m.X = \mathcal{T}(0).X = 0_{n\times n}X = 0_{n\times 1}$ since $\mathcal{T}$ is a ring homomorphism.
$\therefore \lambda^m.X = 0_{n\times 1} \Longrightarrow \lambda^m = 0_{n\times 1}$ (since K has no zero divisors) $\Longrightarrow \lambda = 0$.

(ii) Let $\beta \in KG$ with $\beta^n = 1$. Let λ be an eigenvalue of $(\mathcal{T}(\beta))$ i.e. $(\mathcal{T}(\beta))X = \lambda X$. Now $(\mathcal{T}(\beta))^n.X = \lambda^n.X$. $(\mathcal{T}(\beta))^n.X = \mathcal{T}(\beta^n).X = \mathcal{T}(1).X = I_{n\times n}.X = X$. $\therefore \lambda^n.X = X \Longrightarrow \lambda^n = 1$ (since K is a field) $\Longrightarrow \lambda$ is an n^{th} root of unity.

(iii) Let $f(\gamma) = 0 \; \forall \; \gamma \in KG$ and $\exists \, f \in K[x]$. Let λ be an eigenvalue of $(\mathcal{T}(\gamma))$ $\therefore (\mathcal{T}(\gamma))X = \lambda X$. $\Longrightarrow f(\mathcal{T}(\gamma)).X = f(\lambda).X$ since $\mathcal{T}$ is a $K-linear$ ring homomorphism on RG. $f(\mathcal{T}(\gamma)).X = \mathcal{T}(f(\gamma)).X = \mathcal{T}(0).X = 0.X = 0$.
$\therefore f(\lambda).X = 0 \Longrightarrow f(\lambda) = 0$. ■

Example 3.6 *Let R be a ring and let G be a finite group. We define the* ***trivial group*** *representation of G as :*

$$\mathcal{T} : G \longrightarrow GL_n(R) \qquad g \mapsto I_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

$\mathcal{T}(gh) = I_{n\times n}$. $\mathcal{T}(g)\mathcal{T}(h) = I_{n\times n}.I_{n\times n} = I_{n\times n}$. So $\mathcal{T} : G \longrightarrow \{I_{n\times n}\} \cong C_1$ is a group epimorphism.