Note

$$
\begin{aligned}
& a(\lambda_1.1 + \lambda_2.a + \lambda_3.a^2) \\
\longleftrightarrow & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix} \right) \\
= & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_3 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \\
& \longleftrightarrow \lambda_3.1 + \lambda_1.a + \lambda_2.a^2)
\end{aligned}
$$

We can extend the definition of a left regular group representation to a left regular group ring representation as follows :

Let R be a commutative ring and G a finite group. Define

$$\mathcal{T} : RG \longrightarrow M_n(R), \quad \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g \mathcal{T}_g$$

where $\mathcal{T}_g$ acts on the basis $G = \{g_1 = 1, g_2, \ldots, g_n\}$ by left multiplication (i.e. $\mathcal{T}_g(g_i) = gg_i$.

Lemma 3.4 *$\mathcal{T}$ above is a ring (write $\mathcal{T}_\alpha = \mathcal{T}(\alpha)$) homomorphism from the group ring RG to the set of $n \times n$ matrices over R. Also $\mathcal{T}(r\alpha) = r\mathcal{T}(\alpha)\ \forall\ r \in R$, $\forall\ \alpha \in RG$. Also if R is a field then $\mathcal{T} : RG \longrightarrow M_n(R)$ is injective.*

Proof. Homework 2. ■

If R is commutative then define

- det(α) =det$(\mathcal{T}(\alpha))$
- tr(α) =tr$(\mathcal{T}(\alpha))$
- eigenvalue of (α) = eigenvalue of $(\mathcal{T}(\alpha))$
- eigenvectors of (α) = eigenvectors of $(\mathcal{T}(\alpha))$ where $\alpha \in RG$.