Chapter 3

Group Ring Representations

Definition 3.1 *Let G be a finite group and R a ring. The R-module RG (the group ring RG) with the natural multiplication $g\alpha$ ($g \in G,\ \alpha \in RG$). Now given $g \in G$, g acts on the basis of RG by left multiplication and permutes the basis elements. Define $\mathcal{T} : G \longrightarrow GL_n(R)$ where $g \mapsto \mathcal{T}_g$ and $\mathcal{T}_g$ acts on the basis elements by left multiplication. So if $G = \{g_1 = 1, g_2, \ldots, g_n\}$ and $\mathcal{T}_g\ g_i = gg_i \in G$. The function $\mathcal{T}$ from G to $GL_n(R)$ is called the (left-regular)* ***group representation*** *of the finite group G over the ring R.*

Think of $\mathcal{T}_g$ as left multiplication by a group element or left multiplication of a column vector by a $n \times n$ matrix.

Lemma 3.2 *Let G be a finite group of order n. Let R be a ring. Then the group representation $\mathcal{T}$ is an injective homomorphism (monomorphism) from G to $GL_n(R)$.*

Proof. Let $g, h \in G$ and $g_i \in G$ where g_i are the basis elements. We want to show $\mathcal{T}(gh) = \mathcal{T}(g)\mathcal{T}(h)$. Now $\mathcal{T}(gh).(g_i) = (gh).g_i = g(hg_i) = \mathcal{T}_g(\mathcal{T}_h(g_i))\ \forall g_i \in G = \mathcal{T}(g)\mathcal{T}(h)(g_i)$. $\therefore \mathcal{T}(gh) = \mathcal{T}(g)\mathcal{T}(h)$.

1-1 : We must show that if $\mathcal{T}(g) = I_n \in GL_n(R) \Longrightarrow g = 1_G$. Let $g \in G$ with $\mathcal{T}(g) = I_n$. Then $\mathcal{T}(g)(g_i) = g_i\ \forall g_i \in G$. In particular (with $g_i = g_1 = 1_G$), $\mathcal{T}(g)(1) = I_n \Longrightarrow g.1 = 1 \Longrightarrow g = 1$. ■

Example 3.3 *Let $G = C_3 =< a \mid a^3 = 1 >$.*