**Proposition 2.22** *The set* $\{g-1 \mid g \in G,\ g \neq 1\}$ *is a basis for* $\Delta(G)$ *over* $R$.

i.e. $\Delta(G) = \{\sum_{g\in G} a_g(g-1) \mid g \in G,\ g \neq 1\}$ and the $g-1$ are linearly independant over $R$.

**Proof.** Let $\alpha = \sum_{g\in G} a_g g \in \Delta(G)$. So $\sum_{g\in G} a_g = 0$. Thus $\alpha = \sum_{g\in G} a_g g - 0 = \sum_{g\in G} a_g g - \sum_{g\in G} a_g = \sum_{g\in G} a_g(g-1)$ so this is a spanning set for $\Delta(G)$. We will show linear independance :

Let $\sum_{g\in G} a_g(g-1) = 0$. Then $0 = \sum_{g\in G} a_g g - \sum_{g\in G} a_g = \sum_{g\in G} a_g g = 0 \iff a_g = 0\ \forall\ g \in G$. Since $G$ is linear independant over $R$, by the definition of the group ring $RG$.

■

**Note :** $RG$ has dimension $|G|$ over $R$. $\Delta(G)$ has dimension $|G|-1$ over $R$. If $R$ is a field then these are vector spaces. Otherwise they are $R$-modules.

**Proposition 2.23** *Let* $R$ *be a commutative ring. The map*

$$* : RG \longrightarrow RG \quad \textit{where} \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g g^{-1}$$

*is an* ***involution****. Then* $*$ *has the following properties :*

*(i)* $(\alpha+\beta)^* = \alpha^* + \beta^*$

*(ii)* $(\alpha\beta)^* = \alpha^*\beta^*$

*(iii)* $(\alpha^*)^* = \alpha$

**Proof.** Homework 2. ■

**Proposition 2.24** *Let* $I \lhd R$ *and let* $G$ *be a group. Then*

$$IG = \{\sum_{g\in G} a_g g \mid a_g \in I\} \lhd RG$$