Lemma 2.19 *Let R be s ring. R is semisimple iff every left ideal of R is a direct summand of R.*

Example 2.20 *In the above example $L_1 \oplus L_2$ is a left ideal of R and $(L_1 \oplus L_2) \oplus (L_3 \cdots \oplus L_n) = R$.*

Theorem 2.21 *Let R be a ring. R is semisimple iff every left ideal of R is of the form $L = Re$, where $e \in R$ is an idempotent.*

Proof. ($\Rightarrow$) Assume that R is semisimple. Let $L \overset{l}{\triangleleft} R$. By the previous lemma, L is a direct summand of R. So there exists a left ideal $L' \overset{l}{\triangleleft} R$ such that $L \oplus L' = R$. So $1 = x + y$ for some $x \in L$ and $y \in L'$. (**Question :** Is this decomposition unique ?).
Then $x = x.1 = x(x+y) = x^2+xy$ So $\underbrace{xy}_{\in L'} = \underbrace{x - x^2}_{\in L}$. Thus $xy \in L \cap L' = \{0\}$.
Thus $xy = 0 = x - x^2$, so $x = x^2$. Hence, x is an idempotent. We have shown $L = Rx$ where $x \in L$ so $Rx \subset L$. We must show $L \subset Rx$. Let $a \in L$. Then $a = a.1 = a(x+y) = ax + ay = a$. $\therefore \underbrace{a - ax}_{L} = \underbrace{ay}_{L'} \in L \cap L' = \{0\}$. So $a - ax = 0$ so $a = ax \in Rx$. Thus $L \subset Rx$. So $L = Rx$.

($\Leftarrow$) assume that every left ideal of R is of the form $L = Re$ for some idempotent $e \in R$. We will show that every left ideal is a direct summand of R. Let $L \overset{l}{\triangleleft} R$. Then $L = Re$. Let $L' = R(1-e)$. Then L' is a left ideal of R. (Note $(1-e)^2 = 1 - e - e + e^2 = 1 - 2e + e = 1 - e$). We must show that $L \oplus L; = R$ (i.e. $L + L' = R$ and $L \cap L' = \{0\}$).

Let $x \in R$ Then $x = x.1 = x(e + (1-e)) = xe + x(1-e) \in L + L'$. $\therefore R = L \oplus L'$. Let $x \in L \cap L' = Re \cap R(1-e)$. Then $x = r.e = s(1-e)$, $r, s \in R$. Thus $x.e = (r.e).e = r.e^2 = r.e = x$. Also $x.e = (s(1-e))e = s(e - e^2) = s(0) = 0$. Thus $x = 0$ so $L \cap L' = \{0\}$ and so $R = L \oplus L'$. ∎

Let $\alpha = \sum_{g \in G} a_g g \in RG$. Now all but finitely many of the a_g's are non-zero.
We define the **support of** α as

$$\operatorname{supp} \alpha = \{g \in G \mid a_g \neq 0\}$$

The group $< \operatorname{supp} \alpha >$ (generated by the support of α) is a finitely generated group. So $R < \operatorname{supp} \alpha > \subset RG$.