Example 2.13 *In* $\mathbb{Z}_6$, *3 is an idempotent since* $3^2 = 9 = 3$.

Example 2.14 *In* $M_2(\mathbb{F}_2)$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *are idempotents since*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Definition 2.15 *The* ***center*** *of* R *is*

$$Z(R) = \{z \in R \mid zr = rz \;\forall\; r \in R\}$$

Question : Is $Z(R)$ a ring ?
Question : Is $Z(R)$ an ideal ?

Definition 2.16 e *is called a* ***central idempotent*** *if* $e^2 = e$ *and* $e \in Z(R)$.

Definition 2.17 *A ring* R *is* ***semisimple*** *if it can be decomposed as a direct sum of finitely many minimal left ideals. i.e.* $R = L_1 \oplus \cdots \oplus L_t$, *where* L_i *is a minimal left ideal.*

Note : L is a minimal left ideal of R if L is a left ideal of R ($L \overset{l}{\lhd} R$) and if J is any other left ideal of R contained in L, then either $J = \{0\}$ or $J = L$.

Example 2.18 $M_n(D)$ *is a semisimple ring. Let* $L_1 = \begin{pmatrix} D & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$

and let $L_2 = \begin{pmatrix} 0 & D & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$ *and ... let* $L_n = \begin{pmatrix} 0 & 0 & 0 & \dots & D \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$.

For each i, L_i *is a minimal left ideal of* R *(check!). Also* $M_n(D) = L_1 \oplus \cdots \oplus L_n$ *so* $M_n(D)$ *is semisimple (check!).*