and $B_i = E_{i,h}AE_{k,i}$ so $B_i \in I$. (Now add up all the ideals). Let

$$\begin{aligned} B &= B_1 + B_2 + \cdots + B_n \\ &= a_{h,k}\{E_{1,1} + E_{2,2} + \cdots + E_{n,n}\} \\ &= a_{h,k}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \end{aligned}$$

Thus B is invertible and $B \in I$. Thus (by the secind last lemma)

$$I = M_n(D)$$

■

Definition 2.10 *Let R_1 and R_2 be rings. Define a new ring, the* ***direct sum*** *of R_1 and R_2 as*

$$R_1 \oplus R_2 = \{(r_1, r_2) \,|\, r_1 \in R_1,\ r_2 \in R_2\} \quad (= \underbrace{R_1 \times R_2}_{\textit{cartesian product}})$$

Let (r_1, r_2) and $(s_1, s_2) \in R_1 \oplus R_2$. Define $(r_1, r_2) + (s_1, s_2) = (r_1 + s_1, r_2 + s_2)$ and $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$. This defines a ring (check!).

$R_1 \oplus R_2$ is not a division ring since for any non-zero $r \in R_1$ and $sinR_2$, we have $(r, 0)(0, s) = (r.0, 0.s) = (0, 0) = 0 \in R_1 \oplus R_2$. So $(r, 0)$ and $(0, s)$ are zero divisors. So $(r, 0)$ and $(0, s)$ are not invertible. So Hamilton would not be pleased. We could define $(R_1 \oplus R_2) \oplus R_3 = R_1 \oplus R_2 \oplus R_3$ and $\dots$ and $R_1 \oplus R_2 \oplus \dots \oplus R_3$.

Definition 2.11 *A ring R is called a* ***simple ring*** *if it's only ideals are $\{0\}$ and R (i.e. no non-trivial ideals).*

Note : $M_n(D)$ is a simple ring.

Definition 2.12 *An element $e \in R$ is called an* ***idempotent*** *if $e^2 = e$.*