Proof. Suppose $u \in I$, with u invertible (say $u.v = v.u = 1$). Now since I is an ideal, we have $v.i \in I \; \forall \; i \in I$. In particular, $v.u = 1 \in I$. If r is any element of R, then $r.1 \in I$. So $R \subset I$. So $R = I$ contradiction. ■

Lemma 2.6 *Let D be a division ring. Then*

(i) D has no ideals (apart from $\{0\}$ and itself).

(ii) D has no zero divisors (done before !).

Proof. (i) Let $I \lhd D$, with $I \neq \{0\}$. Let $x \neq 0$ and $x \in I$. So $0 \neq x \in D$, so x is invertible, by the previous lemma $I = D$.

(ii) Let $u.v = 0$ with $u \neq 0$ and $v \neq 0$ (and $u, v \in D$). Now u^{-1} and v^{-1} exists so $u^{-1}(uv) = u^{-1}.0 \Longrightarrow v = 0$, which is a contradiction. ■

Definition 2.7 *An elementary matrix $E_{i,j}$ is the matrix of all whose entries are) except for the $(i, j)^{th}$ entry which is 1.*

Example 2.8

$$E_{1,2} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Lemma 2.9 *Let D be a division ring and $R = M_n(D)$ ($n \times n$ matrices over division ring D). Then $M_n(D)$ has no ideals (apart from $\{0\}$ and $M_n(D)$).*

Proof. If $n = 1$, then this just part (i) of the above lemma. Let $B_i = E_{i,h}AE_{k,i}$. Now all entries of B_i equal) except for the $(i, i)^{\text{th}}$, which is $a_{h,k}$. Thus $B_i = a_{h,k}E_{i,i} \; \forall \; i \in \{1, 2, \dots, n\}$. Now I was a (two sided) ideal, $A \in I$