Definition 1.23 $(\mathbb{Z}_n, +, \cdot)$ *is the ring of integers modulo* n *(where* $n \in \mathbb{Z}$, $n > 0$ *). In fact this is a commutative ring.*

Example 1.24 *Consider* $(\mathbb{Z}_5, +, \cdot)$ *:* $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$ *and* $4^{-1} = 4$. *So* $\mathbb{Z}_5$ *is a division ring, so it is a field.*

Example 1.25 *Consider* $(\mathbb{Z}_6, +, \cdot)$ *:* $1^{-1} = 1$, 2^{-1} *doesn't exist,* 3^{-1} *doesn't exist,* 4^{-1} *doesn't exist and* $5^{-1} = 5$. *So* $\mathcal{U}(\mathbb{Z}_6) = \{1, 5\} = < 5 > \cong C_2$. *So* $\mathbb{Z}_6$ *is not a division ring and hence it is not a field.*

Definition 1.26 *In a ring* R, *if* $a.b = 0$ *but* $a \neq 0$ *and* $b \neq 0$ *then* a *and* b *are called* ***zero divisors****.*

Definition 1.27 *If a ring* R *has no zero-divisors, then* R *is called an* ***integral domain*** *(or just a domain).*

Example 1.28 $(\mathbb{Z}, +, \cdot)$ *is an integral domain since* $a.b = 0 \Longrightarrow a = 0$ *or* $b = 0$.

Example 1.29 *In* $\mathbb{Z}_6$, *2.3=0. So 2 and 3 are zero divisors. Therefore* $\mathbb{Z}_6$ *is not an integral domain.*

Example 1.30 $(\mathbb{Z}_5, +, \cdot)$ *is an integral domain.*

Lemma 1.31 *Every division ring is an integral domain.*

Proof. We assume that R is a division ring. We want to show that R has no zero divisors. Proceed by contradiction : Assume $a.b = 0$, where $a \neq 0$ and $b \neq 0$. Since $0 \neq a \in R$ then we have $a^{-1} \in R$. $\therefore a^{-1}(ab) = a^{-1}(0) = 0 = (a^{-1}a)b = 1.b = b = 0$. This is a contradiction. ■

Notes :

(1) The converse is not true. i.e. there are integral domains which are not division rings. e.g. $(\mathbb{Z}, +, \cdot)$ is not an integral domain but not a division ring.

(2) Zero-divisors are never invertible.

Example 1.32 *Let* $R = \mathbb{F}_2 = \mathbb{Z}_2$ *and* $G = C_2$ *(*$\mathbb{Z}_2$ *is the ring of order 2, which is a field). Writing down the elements :* $\mathbb{F}_2 = \{0, 1\}$ *and* $C_2 = \{1, x\} = < x > = < x \mid x^2 = 1 >$.