If M is a left R-module and a right R-module, then we call M a ***(two-sided) R-module****.*

Definition 1.13 *Let R be a ring. An element $a \in R$ is* ***invertible*** *in R if $\exists\, b \in R$ such that $a.b = b.a = 1$.*

We write $b = a^{-1}$ (the inverse of a) and say that a is a **unit** of R.

Definition 1.14

$$\mathcal{U}(R) = \{a \in R \mid \textit{if } a \textit{ is a unit of } R\}$$

Note that $\mathcal{U}(R)$ is a group (with multiplication) called the **group of units** of R.

Example 1.15 $\mathcal{U}(\mathbb{Z}) = \{+1, -1\}$, *the cyclic group of order 2 (written C_2).*

Example 1.16 $\mathcal{U}(\mathbb{Q}) = \mathbb{Q} \setminus \{0\}$.

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \textit{ where } a \neq 0,\ b \neq 0$$

Example 1.17 $\mathcal{U}(\mathbb{R}) = \mathbb{R} \setminus \{0\}$.

Example 1.18 $\mathcal{U}(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

Example 1.19 $\mathcal{U}(\mathbb{H}) = \mathbb{H} \setminus \{0\}$.

Example 1.20 $\mathcal{U}(M_n(\mathbb{R})) = \{A \in M_n(\mathbb{R}) \mid det\, A \neq 0\} = GL_n(\mathbb{R})$.

Definition 1.21 *A ring R is called a* ***division ring*** *if every non-zero element of R is a unit. i.e. $\mathcal{U}(R) = R \setminus \{0\}$.*

Note : $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are division rings. $\mathbb{Z}$ and $M_n(\mathbb{R})$ are not division rings.

Definition 1.22 *A division ring R is called a* ***(commutative) field*** *if R is a commutative ring.*

Note : $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields. $\mathbb{H}$ is not a field (non-commutative). $\mathbb{Z}$ is not a field (not a division ring).