$$i^2 = j^2 = k^2 = -1 = ijk$$

$$ij = k \qquad ji = -k$$

$$jk = i \qquad kj = -i$$

$$\boldsymbol{ki = j} \qquad ik = -j$$

$1.i = i.1 = i,\ 1.j = j.1 = j,\ 1.k = k.1 = k$ *and* $1.1 = 1$

Now define:

$$(a + bi + cj + dk)(e + fi + gj + hk) = (ae - bf - cg - dh) + (af + be + ch - dg)i$$
$$(ag + ce - bh + df)j + (ah + de + bg - cf)k$$

This multiplication gives us a non-commutative ring ($ij \neq ji$), *called the Quaternions* ($\mathbb{H}$).

Example 1.9 (***1840's Hamilton***) *Consider an n-dimensional vector space (over* $\mathbb{R}$ *say) with basis* $\{e_1, e_2, \ldots, e_n\}$ *(the basic units). Define the product* $e_i.e_j\ \forall\ i, j = 1\ldots n$. *Then (as in the previous example) insist on the distributive laws and we see that this new object is a ring, called the set of **Hypercomplex Numbers*** (M).

Example 1.10 *If* $\{e_1, e_2, \ldots, e_n\}$ *forms a group (under multiplication)* G, *then the hypercomplex numbers generated by* G *is called the **Group Ring*** ($\mathbb{R}G$). ***Arthur Cayley 1854.***

Definition 1.11 *Given a group* G *and a ring* R, *define the **Group Ring*** RG *to be the set of all linear combinations*

$$\alpha = \sum_{g \in G} a_g g$$

where $a_g \in R$ *and where only finitely many of the* $a_g{}^s$ *are non-zero.*
Define the sum

$$\alpha + \beta = \left(\sum_{g \in G} a_g g\right) + \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} (a_g + b_g) g.$$