Definition 1.5 *If* $\exists 1 \in R$ *such that* $1.a = a.1 \;\forall\; a \in R$, *then R is a* ***ring with identity***. *Otherwise R is a ring without identity.*

For us, R (usually) is a ring with identity.

Example 1.6 *The set* $M_n(\mathbb{R})$ *of all* $n \times n$ *matrices with real coefficients is a ring (with matrix addition and matrix multiplication).*

(i) $A + (B + C) = (A + B) + C \quad \checkmark$

(ii) *Let* $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *then* $0 + A = A + 0 = A \quad \checkmark$

(iii) *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,*then* $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ *and* $-A + A = A + -A = 0 \quad \checkmark$

(iv) $A + B = B + A \quad \checkmark$

(v) $A.(B.C) = (A.B).C \quad \checkmark$

(vi) $A.(B + C) = A.B + B.C \quad \checkmark$

(vii) $(A + B).C = A.C + B.C \quad \forall\; A, B, C \in M_n(\mathbb{R}) \quad \checkmark$

Note : $M_n(\mathbb{R})$ is a non-commutative ring (since $AB \neq BA \;\forall\; A, B \in M_n(\mathbb{R})$).

Example 1.7 $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ *is a ring (the complex numbers). It is also a 2-dimensional vector space over* $\mathbb{R}$ *with basis* $\{1, i\}$.

Example 1.8 *Consider a 4-dimensional vector space over* $\mathbb{R}$ *with basis* $\{1, i, j, k\}$. *We define multiplication as follows*