Now let $\alpha = \left(\sum_{g\in G} a_g g\right)$ and $\beta = \left(\sum_{h\in G} b_h h\right)$.

$$\varepsilon(\alpha\beta) = \left(\sum_{g,h\in G} a_g b_h gh\right) = \sum_{g,h\in G} a_g b_h$$

$$\varepsilon(\alpha)\varepsilon(\beta) = \varepsilon\left(\sum_{g\in G} a_g g\right)\varepsilon\left(\sum_{h\in G} b_h h\right) = \left(\sum_{g\in G} a_g\right)\left(\sum_{h\in G} b_h\right) = \sum_{g,h\in G} a_g b_h$$

$\therefore \varepsilon(\alpha+\beta) = \varepsilon(\alpha)\varepsilon(\beta)$ and ε is a ring homomorphism.

$Ker(\varepsilon) = \{\alpha = \sum_{g\in G} a_g g \,|\, \varepsilon(\alpha) = \sum_{g\in G} a_g = 0\}$. $Ker(\varepsilon)$ is non empty and non trivial.

Example 2.2 *$rg + (-rh) \in Ker(\varepsilon)$ since $\varepsilon(rg + (-rh)) = r - r = 0$.*

Now $\dfrac{RG}{Ker(\varepsilon)} \cong R$. $Ker(\varepsilon)$ is an ideal called the **augmentation ideal** of RG and is denoted by $Ker(\varepsilon) = \Delta(RG)$.

Let $u \in \mathcal{U}(RG)$. Say $u.v = v.u = 1$. Then $\varepsilon(uv) = \varepsilon(1) = 1 = \varepsilon(u)\varepsilon(v) = 1 \in R$. So $\varepsilon(u)$ is invertible in R, with inverse $\varepsilon(v)$. So $\varepsilon(\mathcal{U}(RG)) \subset \mathcal{U}(R)$ i.e. ε sends units of RG to units of R.

Let $u \in \mathcal{ZD}(RG)$. Say $u.v = v.u = 0$ where $u, v \neq 0$. Then $\varepsilon(uv) = \varepsilon(u)\varepsilon(v) = \varepsilon(0) = 0$. Thus $\varepsilon(u)\varepsilon(v) = 0$. So either $\varepsilon(u) = 0$ or $\varepsilon(v) = 0$ or $\varepsilon(u)$ and $\varepsilon(v)$ are zero divisors in R.

If R has no zero divisors then this forces $\varepsilon(u) = 0$ or $\varepsilon(v) = 0$.

Example 2.3 *List all the elements of $\mathbb{F}_3C_2$, $\mathcal{U}(\mathbb{F}_3C_2)$ and $\mathcal{ZD}(\mathbb{F}_3C_2)$.*

$C_2 = \{1, x\}$ and $\mathbb{F}_3 = \{0,1,2\}$.$\mathbb{F}_3C_2 = \{a_1.1 + a_2.x \,|\, a_i \in \mathbb{F}_3\}$. Thus $|\mathbb{F}_3C_2| = 3.3 = 3^2 = 9$ ($|\mathbb{F}_3|^{|C_2|}$).