Example 1.42 *Consider the ring* $(\mathbb{Z}_5, +, \cdot)$. *Let* $I_2 = \{2.a : a \in \mathbb{Z}_5\} = \{0, 2, 4, 1, 3\} = \mathbb{Z}_5$. *Therefore the only ideals of* $\mathbb{Z}_5$ *are* $\{0_{\mathbb{Z}_5}\}$ *and* $\mathbb{Z}_5$. *i.e. Let* $I \lhd \mathbb{Z}_5$, *then* $|I|/|\mathbb{Z}_5|$ *so* $|I| = 1$ *or* 5 *so* $I = \{0_{\mathbb{Z}_5}\}$ *or* $\mathbb{Z}_5$

Let $f : R \longrightarrow S$ be a ring homomorphism, then $f(1_r) = 1_s$ is not necessarily true.

Example 1.43 *Define* $f : M_2(\mathbb{Q}) \longrightarrow M_3(\mathbb{Q})$ *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then $f\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* f *is a ring homomorphism. However*

$f(I_2) = f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3$.

Note that here $f(A)f(I_2) = f(a.I_2) = f(A)$. So $f(I_2)$ seems to work like the multiplicative identity on the range of f.

Let $f : R \longrightarrow S$ be a ring homomorphism. Then $Ker(f) = \{x \in R : f(x) = 0\}$. If $x, y \in Ker(f)$, then $f(x + y) = f(x) + f(y) = 0 + 0 = 0$. Also $f(x - y) = f(x) - f(y) = 0 - 0 = 0$.

Let $x \in Ker(f)$, $s \in R$. Is $xs \in Ker(f)$? $f(xs) = f(x)f(s) = 0.f(s) = 0$. $\therefore xs \in Ker(f)$. So $Ker(f)$ is an ideal of R.

Definition 1.44 *A ring homomorphism* $f : R \longrightarrow S$ *is called*

(i) a ***monomorphism*** *(or embedding) if* f *is injective.*

(ii) an ***epimorphism*** *if* f *is surjective.*

Example 1.45 $\mathbb{Z} \overset{f}{\hookrightarrow} \mathbb{Q}$ *where* $f(n) = n$. $Ker(f) = \{0\} \subset \mathbb{Z}$.

Example 1.46 $\mathbb{Z} \overset{g}{\rightarrowtail} 2\mathbb{Z}$ *where* $g(n) = 2n$. $Ker(g) = \{0\} \subset \mathbb{Z}$.

Example 1.47 *Let* p *be a prime number. Define* $f : \mathbb{Z} \longrightarrow \mathbb{Z}_p$ *by* $f(n) = n + p\mathbb{Z}$.