Similarly we could define a right ideal of $R$. If $L$ is a left ideal of $R$ and a right ideal of $R$, we say that $L$ is a **two-sided** ideal of $R$.

*** (used in the same way that normal subgroups are used in group theory). i.e. If $N \lhd G \Longrightarrow G \longrightarrow \frac{G}{N}$, $g \mapsto g.N$ is a group homomorphism with kernal $N$ and image $\frac{G}{N}$, the factor group or quotient group of $G$ by $N$.

$$\frac{G}{N} = \{gN \ : \ g \in G\}.$$

**Recall :** $1^{\text{st}}$, $2^{\text{nd}}$ and $3^{\text{rd}}$ isomorphism theorems of groups.

Let $I$ be an ideal of $R$. We write $I \lhd R$. Notice that $I$ is a ring (usually without the multiplicative identity $1_r$). $\Longrightarrow I$ is a subring of $R$.

**Example 1.39** *Consider the ring $(\mathbb{Z}, +, \cdot)$. Let $n \in \mathbb{Z}$. Then $I = n\mathbb{Z} = \{n.a \ : \ a \in \mathbb{Z}\}$ is a (two sided) ideal of $\mathbb{Z}$, since*

$$na - nb = n(a - b) \in n\mathbb{Z} \, \forall \, a, b \in \mathbb{Z}$$
$$c(n.a) = n(c.a) \in n\mathbb{Z} \, \forall \, c \in \mathbb{Z}$$

**Example 1.40** *Consider the ring $(\mathbb{Z}_6, +, \cdot)$. What are the ideals of $(\mathbb{Z}_6, +, \cdot)$ ? Now consider the subset $I_2 = \{2.a \ : \ a \in \mathbb{Z}_6\} = \{0, 2, 4\}$. $I_2$ is an ideal of $\mathbb{Z}_6\}$ (exercise). $I_3 = \{3.a \ : \ a \in \mathbb{Z}_6\} = \{0, 3\}$ is an ideal of $\mathbb{Z}_6\}$ (exercise). $0 = \{0_{\mathbb{Z}_6}\} \lhd \mathbb{Z}_6\}$. Also $\mathbb{Z}_6 \unlhd \mathbb{Z}_6$. Note that $\mathbb{Z}_6\}$ is the only ideal of $\mathbb{Z}_6\}$ which contains $1_{\mathbb{Z}_6}$. Note : $I_1 = \{1.a \ : \ a \in \mathbb{Z}_6\} = \mathbb{Z}_6$. Are there any more ideals of $\mathbb{Z}_6$ ? Let $I$ be an ideal of $\mathbb{Z}_6$. What is the size of $I$ ?*

**Lemma 1.41** ( *Langrange theorem for rings* ) *Let $I$ be an ideal of a finite ring $R$. Then $|I| \, / \, |R|$.*

**Proof.** $(R, +)$ is a group, $(I, +)$ is a subgroup. Apply Lagranges theorem (for groups), we get $|I| \, / \, |R|$. ■

Applying this lemma to the previous example, we see that $|I| = 1$, 2, 3 or 6. If $|I| = 1$, then $I = \{0_{\mathbb{Z}_6}\}$. If $|I| = 6$, then $I = \mathbb{Z}_6$. If $|I| = 2$, then $I = \{0, 3\}$. If $|I| = 3$, then $I = \{0, 2, 4\}$. Thus $\mathbb{Z}_6$ has 4 ideals.