14. Compactness

(14.1)**Definition**: Let $F = \{A_\lambda\}_{\lambda\in\Lambda}$ is a family of subsets in X, and let $A \subseteq X$. We said that F is a covering of A, if $A \subseteq A_\lambda$. If Λ is a finite, then F is a finite covering of A.

(14.2)**Example:** Let $X = \{1,2,3,4,5\}, A = \{1,2\}$, then

1. A family $\{\{1\},\{2,3\}\}$ represents a covering of A, since $\{1,2,3\} = \{1\} \cup \{2,3\} \Longrightarrow A \subseteq \{1\} \cup \{2,3\}$.
2. A family $\{\{2\},\{4,5\}\}$ does not represent a covering of A, since $A \nsubseteq \{\{2\} \cup ,\{4,5\}\}$.
3. A family $\{\{1,2\},\{3,4\},\{1,3,5\}\}$ represents a covering of A and X.

(14.3)**Example:**

1. A family $F = \{\left[1-\frac{1}{n},\frac{1}{n}\right]: n \in \mathbb{Z}^+\}$ represents an infinite covering of $A = (0,1)$.
2. A family $F = \{(n, n+3): n \in \mathbb{Z}\}$ represents an infinite covering of $\mathcal{R}$.
3. A family $F = \{(n, n+1): n \in \mathbb{Z}^+\}$ does not represent a covering of $\mathcal{R}$.

(14.4)**Definition**: Let $A \subseteq X$, $F = \{A_\lambda\}_{\lambda\in\Lambda}, G = \{B_\gamma\}_{\gamma\in\Lambda'}$ are covering of A, we said that F is a sub covering from G, if for all $\lambda \in \Lambda\ \exists \gamma \in \Lambda' \ni A_\lambda = B_\gamma$.

(14.5)**Example:** Each of $F = \{(n, n+3): n \in \mathbb{Z}\}, G = \{(r, r+3): r \in \mathcal{R}\}$ are covering of $\mathcal{R}$ and F is a subfamily of G.

(14.6)**Definition**: Let A is a subset of (X, d) and let $F = \{A_\lambda\}_{\lambda\in\Lambda}$ is a covering of A. We said that F is an open cover, if A_λ is an open set in X $\forall \lambda \in \Lambda$.

(14.7)**Example**: In $(\mathcal{R}, d_u)$. Prove that a family $F = \{\left(\frac{1}{n},2\right): n \in \mathbb{Z}^+\}$ is an open cover of $A = (0,1)$.

Solution: let $x \in A \Longrightarrow 0 < x < 1$,

Since $x > 0$ (by Archimedes property) $\Longrightarrow \exists k \in \mathbb{Z}^+ \ni \frac{1}{k} < x$.

Since $x < 1 \Longrightarrow \frac{1}{k} < x < 2 \Longrightarrow x \in (\frac{1}{k}, 2) \Longrightarrow x \in \bigcup_{n\in\mathbb{Z}^+}(\frac{1}{n}, 2)$

$\Longrightarrow A \subset \bigcup_{n\in\mathbb{Z}^+}(\frac{1}{n}, 2) \Longrightarrow F$ is a covering of A.

Since $(\frac{1}{n}, 2)$ is open set $\forall n \in \mathbb{Z}^+ \Longrightarrow F$ is an open set of A.

(14.8)**Example**: In $(\mathcal{R}, d_u)$, we have

$F_1 = \{(-n, n): n \in \mathbb{Z}^+\}$, $F_2 = \{(-3n, 3n): n \in \mathbb{Z}^+\}$,$F_3 = \{(2n-1, 2n+1): n \in \mathbb{Z}\}$ are an open cover of $\mathcal{R}$, also F_2 is a sub cover of F_1.

(14.9)**Example**: Let (X, d) be discrete metric space and $A \subseteq X$. Prove that $F = \{\{x\}: x \in A\}$ is an open cover of A.

Solution: since $A = \bigcup_{x\in A}\{x\} \Longrightarrow F$ is a covering of A.

Since (X, d) is discrete metric space $\Longrightarrow \{x\}$ an open set in X $\forall x \in X$

$\Longrightarrow F$ is an open cover of A.

(14.10)**Definition**: Let (X, d) is metric space and let $A \subseteq X$. We said that A is a compact set in X, if for all open cover A contains a finite sub covering.

(14.11)**Example**: In $(\mathcal{R}, d_u)$, we have

1. $A = (0,1)$does not compact in $\mathcal{R}$.
2. $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots, 0\}$ is a compact in $\mathcal{R}$.
3. A space $\mathcal{R}$ does not compact.

Solution: (1) Take $F = \{(\frac{1}{n}, 2): n \in \mathbb{Z}^+\}$ is an open cover of A, but F does not contain on a finite sub cover $\Longrightarrow A$ does not compact.

(14.12) **Example**: Every indiscrete metric space is a compact, since an unique open cover of X is X.

(14.13) **Theorem:** Every finite set in a metric space is a compact.

Proof: let A is a finite set in $(X, d) \Longrightarrow A = \{a_1, a_2, \dots, a_n\}$

Let $F = \bigcup_{\lambda\in\Lambda} G_\lambda$ is a open cover of A in X.

$\Longrightarrow A \subseteq \bigcup_{\lambda\in\Lambda} G_\lambda$, $G_\lambda$is an open set in X $\forall \lambda \in \Lambda$.

Since $a_i \in A$ $\forall i = 1,2, \dots, n$

$$\Longrightarrow a_i \in \bigcup_{\lambda\in\Lambda} G_\lambda \ \forall i = 1,2, \dots, n$$

$\Longrightarrow \forall i \; \exists \; \lambda_i \in \Lambda \ni a_i \in G_{\lambda_i}$

$\Longrightarrow \{G_{\lambda_1}, G_{\lambda_2}, \dots, G_{\lambda_n}\}$ is a finite sub covering from F of A.

$\Longrightarrow A$ is a compact set.

(14.14)**<u>Example</u>**: Let (X, d) is discrete metric space, then X is a compact $\Leftrightarrow X$ is a finite.

(14.15)**<u>Theorem</u>**: Let (Y, d_Y) is a subspace of a metric space (X, d) and $A \subset Y$, then A is a compact in $X \Leftrightarrow A$ is a compact in Y.

(14.16)**<u>Theorem</u>**: Every closed set in a compact metric space is a compact.

(14.17)**<u>Theorem</u>**: Every compact set in a metric space is a closed and bounded.

(14.18)**<u>Definition:</u>** We said that a family of sets that satisfy a finite intersection property, if intersection every finite subfamily is a non- empty set.

(14.19)**<u>Theorem;</u>** A metric space (X, d) is a compact $\Leftrightarrow$ if every family of a closed sets satisfies a finite intersection property, then its non-empty set.

(14.20)**<u>Definition:</u>** We said that a metric space (X, d) is a countable compact, if for all open cover and countable in X contains on a finite sub covering.

(14.21)**<u>Theorem;</u>** A metric space (X, d) is a countable compact $\Leftrightarrow$ every countable family of a closed sets and satisfy a finite intersection property is a non- empty intersection.