13. Uniform Continuity

(13.1)**Definition**: If $(X, d_1), (Y, d_2)$ be metric spaces. We said that a function $f: X \to Y$ is an uniform continuous on X, if $\forall \varepsilon > 0\ \exists \delta > 0 \ni \forall x, y \in X$, then $d_1(x, y) < \delta \Longrightarrow d_2(f(x), f(y)) < \varepsilon$.

(13.2)**Theorem**: Every uniform continuous is continuous.

Proof: let $(X, d_1), (Y, d_2)$ are metric spaces and let a function $f: X \to Y$ is an uniform continuous. Let $x_0 \in X$, we must prove that f be continuous at x_0.

Let $\varepsilon > 0$, since f is an uniform continuous $\Longrightarrow \exists \delta > 0 \ni \forall x, y \in X$, then

$d_1(x, y) < \delta \Longrightarrow d_2(f(x), f(y)) < \varepsilon$, since $x_0 \in X \Longrightarrow \forall x \in X \Longrightarrow d_1(x, x_0) < \delta \Longrightarrow d_2(f(x), f(x_0)) < \varepsilon \Longrightarrow f$ is a continuous at $x_0 \Longrightarrow f$ is a continuous.

(13.3)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space and a function $f: \mathcal{R} \to \mathcal{R}$ defined by $f(x) = x^2, x \in \mathcal{R}$, then f is continuous, but does not uniform continuous.

Solution: $\varepsilon > 0 \ni \forall \delta > 0\ \exists x, y \in \mathcal{R}$ and $|x - y| < \delta \Longrightarrow |f(x) - f(y)| > \varepsilon$

Let $\delta > 0$, (by Archimedes property) $\exists k \in \mathbb{Z}^+ \ni \frac{1}{k} < \delta$

Put $y = k + \frac{1}{k}, x = k \Longrightarrow |x - y| = \frac{1}{k} < \delta$, but $|f(x) - f(y)| = 2 + \frac{1}{k^2} > 2$

$\Longrightarrow f$ does not uniform continuous.

Real- Valued Functions

(13.4)**Definition:** Let $f, g \in RV(X) = \{f: X \to \mathcal{R}\}, \lambda \in \mathcal{R}$. Define $f + g, \lambda f, \frac{f}{g}, |f|$ as following:

- $(f + g)(x) = f(x) + g(x)$
- $(\lambda f)(x) = \lambda f(x)$
- $(fg)(x) = f(x)g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0\ \forall x \in X$
- $|f|(x) = |f(x)|$

(13.5)**Theorem:** If $f, g \in C(X)$ which denoted to set of all a continuous functions and defined from (X, d) into $(\mathcal{R}, d_u)$ and $\lambda \in \mathcal{R}$, then

1. $f + g \in C(X)$.
2. $\lambda f \in C(X)$.
3. $fg \in C(X)$.
4. $\frac{f}{g} \in C(X)$.
5. $|f| \in C(X)$.

Proof: (1) let $x_0 \in X, \varepsilon > 0$

Since $f, g \in C(X) \Rightarrow f: X \to \mathcal{R}, g: X \to \mathcal{R}$ are continuous functions

$\Rightarrow f, g$ are continuous at x_0

Since $f: X \to \mathcal{R}$ is continuous at $x_0 \Rightarrow \exists \delta_1 > 0 \ni \forall x \in X \Rightarrow d(x, x_0) < \delta_1 \Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{2}$

Since $g: X \to \mathcal{R}$ is continuous at $x_0 \Rightarrow \exists \delta_2 > 0 \ni \forall x \in X \Rightarrow d(x, x_0) < \delta_2 \Rightarrow |g(x) - g(x_0)| < \frac{\varepsilon}{2}$

Put $\delta = \min\{\delta_1, \delta_2\} \Rightarrow \delta > 0 \; \forall x \in X \Rightarrow d(x, x_0) < \delta$

$(f + g)(x) - (f + g)(x_0) = \big(f(x) + g(x)\big) - (f(x_0) + g(x_0))$

$= \big(f(x) - f(x_0)\big) + (g(x) - g(x_0))$

$$|(f + g)(x) - (f + g)(x_0)| = |f(x) - f(x_0)| + |(g(x) - g(x_0))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow f + g$ is continuous at $x_0 \Rightarrow f + g \in C(X)$.

Boundedness

(13.6)**Definition:** Let (X, d) be metric space and $A \subseteq X$. We said that A is bounded in X, if $\delta(A) = \sup\{d(x, y): x, y \in A\} < \infty$ or $B = \{d(x, y): x, y \in A\}$ is bounded in $\mathcal{R}$. We say that X is bounded space, if $\delta(X) < \infty$.

(13.7)**Theorem:** Let (X, d) be metric space and $A \subseteq X$. We said that A is bounded in $X \Leftrightarrow \forall x_0 \in A \; \exists k \in \mathbb{Z}^+ \ni d(x, x_0) < k \; \forall x \in A$.

(13.8)**Example:** In usual metric space $(\mathcal{R}, d_u)$, we have

1. $A_1 = (a,b), A_2 = (a,b], A_3 = [a,b), A_4 = [a,b]$ be a bounded, since $\delta(A_i) = b - a \; \forall i = 1,2,3,4$.
2. A space $\mathcal{R}$ is unbounded, since $\delta(\mathcal{R}) = \infty$.

(13.9)**Definition:** Let $X, d)$ be metric space and $(\mathcal{R}, d_u)$ is usual metric space. We said that a function $f: X \to \mathcal{R}$ is a bounded, if $\exists \; M \in \mathcal{R}^+ \ni |f(x)| \leq M \; \forall x \in X$.

Intermediate Value Property

(13.10)**Definition**: Let $(\mathcal{R}, d_u)$ is usual metric space. We said that $f: [a,b] \to \mathcal{R}$ satisfies an intermediate value property, if $\forall x, y \in [a,b], \forall s$ between $f(x), f(y) \exists \; z$ between $x, y \ni f(z) = s$.

(13.11)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space and let a function $f: [a,b] \to \mathcal{R}$ defined by $f(x) = x \; \forall \; x \in [a,b]$, then a function f satisfies an intermediate value property.

Solution: let $x, y \in [a,b] \ni x < y$ and let $f(x) < s < f(y)$

Since $f(x) = x \; \forall \; x \in [a,b] \Longrightarrow x < s < y$

Since $f(s) = s \Longrightarrow f$ satisfies an intermediate value property.

(13.12) **Theorem** (**Intermediate Value Theorem**)

Let $(\mathcal{R}, d_u)$ is usual metric space. If a function $f: [a,b] \to \mathcal{R}$ is a continuous, then $\forall \; s$ between $f(a), f(b), \exists \; z$ in $[a,b] \ni f(z) = s$.

(13.13)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space. If a function $f: [0,1] \to \mathcal{R}$ defined as $f(x) = \begin{cases} \sin\frac{1}{x}, 0 < x \leq 1 \\ 0, x = 0 \end{cases}$, then f satisfies an intermediate value property, but its discontinuous.