12. Continuity

(12.1)**Definition**: If $(X, d_1), (Y, d_2)$ be metric spaces. We said that a function $f: X \to Y$ is continuous at $x_0 \in X$, if $\forall$ open set $U \subseteq Y$ contains $f(x_0), \exists$ an open set $V \subseteq X$ contains $x_0 \ni f(V) \subset U$.

(12.2)**Theorem:** Let $(X, d_1), (Y, d_2)$ be metric spaces, then a function $f: X \to Y$ be continuous at point $x_0 \in X \Leftrightarrow \forall$ open ball $B_\varepsilon(f(x_0))$ in Y $\exists$ an open ball $B_\delta(x_0)$ in $X \ni f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0))$.

This means, $\forall \varepsilon > 0 \ \exists \ \delta > 0 \ni \forall x \in X \Longrightarrow d_1(x, x_0) < \delta \Longrightarrow d_2(f(x), f(x_0)) < \varepsilon$.

(12.3)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space. Prove that a function $f: \mathcal{R} \to \mathcal{R}$ defined by $f(x) = x^2, x \in \mathcal{R}$ is a continuous.

Solution: let $x_0 \in \mathcal{R}, \varepsilon > 0$.

$|f(x) - f(x_0)| = |x^2 - {x_0}^2| = |(x - x_0)(x + x_0)| = |x - x_0||x + x_0|$

Since $|x + x_0| \leq |x| + |x_0|$

So, $|f(x) - f(x_0)| \leq |x - x_0|(|x| + |x_0|) \ldots (1)$

Since $x = (x - x_0) + x_0 \Longrightarrow |x| = |(x - x_0) + x_0| \Longrightarrow |x| \leq |x - x_0| + |x_0|$

$\Longrightarrow |x| - |x_0| \leq |x - x_0|$

If $|x - x_0| < 1 \Longrightarrow |x| - |x_0| < 1 \Longrightarrow |x| < 1 + |x_0| \ldots (2)$

From (1), (2), we get

$|f(x) - f(x_0)| \leq |x - x_0|(1 + 2|x_0|) \ldots (3)$

Take $\delta = \min\{1, \frac{\varepsilon}{1+2|x_0|}\}$

Now, let $x \in \mathcal{R} \ni |x - x_0| < \delta \Longrightarrow |x - x_0| < \frac{\varepsilon}{1+2|x_0|}$ and $|x - x_0| < 1$

$\Longrightarrow |x - x_0| < (1 + 2|x_0|) < \varepsilon$

From (3), we get

$|f(x) - f(x_0)| < \varepsilon$

$\Longrightarrow f$ is a continuous at $x_0 \Longrightarrow f$ is a continuous.

(12.4)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space. Prove that a function $f: \mathcal{R} \to \mathcal{R}$ defined by $f(x) = \frac{1}{x}, x \in \mathcal{R}^+$ is a continuous at $x = 2$.

Solution: let $\varepsilon > 0, |f(x) - f(2)| = \left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|2-x|}{2x}$ (since $x > 0$)

If $|x - 2| < 1 \Rightarrow -1 < x - 2 < 1 \Rightarrow 1 < x < 3 \Rightarrow x > 1 \Rightarrow \frac{1}{x} < 1$

$$\Rightarrow \frac{|2-x|}{2x} < \frac{1}{2}|2-x|$$

Choose $\delta = \min\{1, 2\varepsilon\}$

Now, let $x \in \mathcal{R}^+ \ni |x - 2| < \delta \Rightarrow |x - 2| < 2\varepsilon$ and $|x - 2| < 1$

$\Rightarrow \frac{1}{2}|x - 2| < \varepsilon$

$|f(x) - f(2)| < \frac{1}{2}|x - 2| < \varepsilon$

$\Rightarrow f$ is a continuous at $x = 2$.

(12.5)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space. Prove that a function $f: \mathcal{R} \to \mathcal{R}$ defined by $f(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$, is a continuous on $\mathcal{R} \backslash \{0\}$.

(12.6)**Theorem:** Let $(X, d_1), (Y, d_2)$ be metric spaces, and $f: X \to Y$ a function, then the following properties are equivalent:

1. A function f is a continuous.
2. If an open set $G \subset Y$, then $f^{-1}(G)$ be an open set in X.
3. If a closed set $H \subset Y$, then $f^{-1}(H)$ be a closed set in X.
4. $f(\bar{A}) \subseteq \overline{f(A)}\ \forall A \subset X$.
5. $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})\ \forall B \subset Y$.
6. $f^{-1}(B^{\circ}) \subseteq (f^{-1}(B))^{\circ}$.

(12.7)**Theorem:** Let $(X, d_1), (Y, d_2), (Z, d_3)$ be metric spaces, and $f: X \to Y, g: Y \to Z$ be a continuous functions, then a function $g \circ f: X \to Z$ be a continuous function.

Proof: let G is an open set in Z.

Since a function $g: Y \to Z$ is a continuous $\Rightarrow g^{-1}(G)$ is an open set in Y.

Since a function $f: X \to Y$ is a continuous $\Longrightarrow f^{-1}(g^{-1}(G))$ is an open set in X, but $f^{-1}(g^{-1}(G)) = (f^{-1} \circ g^{-1})(G) = (g \circ f)^{-1}(G)$

$\Longrightarrow (g \circ f)^{-1}(G)$ is an open set in X

$\Longrightarrow g \circ f$ be a continuous function.

(12.8)**Example:** Let $(X, d_1), (Y, d_2)$ be metric spaces, and $f: X \to Y$ a function. Prove that

1. If f is a constant, then f is a continuous.
2. If (X, d_1) be discrete, then f is a continuous.
3. If (X, d_1) be indiscrete, then f is a continuous.

Solution: (1) since f is a constant, then $\exists\, b \in Y \ni f(x) = b\ \forall x \in X$.

Let G be an open set in Y.

$$f^{-1}(G) = \begin{cases} \emptyset, b \notin G \\ X, b \in G \end{cases}$$

Since $\emptyset, X$ be an open sets $\Longrightarrow f^{-1}(G)$ be an open set in $X \Longrightarrow f$ is a continuous.

Sequentially Continuity

(12.9)**Definition:** Let $(X, d_1), (Y, d_2)$ be metric spaces. We said a function $f: X \to Y$ be sequentially continuity at $x \in X$, if every sequence $\{x_n\}$ in $X \ni x_n \to x_0 \Longrightarrow f(x_n) \to f(x_0)$ in Y.

(12.10)**Theorem:** Let $(X, d_1), (Y, d_2)$ be metric spaces, then a function $f: X \to Y$ be a continuous at $x_0 \in X \Longleftrightarrow f$ be sequentially continuity at $x_0 \in X$.

(12.11)**Example:** Let $(\mathcal{R}, d_u)$ be usual metric space. Prove that a function $f: \mathcal{R} \to \mathcal{R}$ defined by $f(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$, is a discontinuous at $x = 0$.

Solution: take $x_n = \frac{1}{n} \Longrightarrow \{x_n\}$ in $\mathcal{R}$ and $x_n \to 0$,

since $\frac{1}{n} > 0\ \forall n \in \mathbb{Z}^+ \Longrightarrow f\left(\frac{1}{n}\right) = 1 \Longrightarrow f(x_n) = 1$

$\Longrightarrow f(x_n) \to 1 \Longrightarrow f(x_n) \nrightarrow f(0) = 0$

$\Longrightarrow f$ is a discontinuous at $x = 0$.