11. Some Important Metric Spaces

(11.1) **Lemma**: If $a \geq 0, b \geq 0, p > 1, q > 1 \ni \frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$ and $ab = \frac{1}{p} a^p + \frac{1}{q} b^q \Leftrightarrow a^p = b^q$.

(11.2) **Theorem**: (Holders Inequality)

If $p, q \in \mathcal{R} \ni \frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow \sum_{i=1} |x_i y_i| \leq (\sum_{i=1} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1} |y_i|^q)^{1/q} \ni x_i, y_i \in \mathcal{R}$.

(11.3) **Corollary**: (**Cauchy-schwars Inequality**)
$\sum_{i=1} |x_i y_i| \leq (\sum_{i=1} |x_i|^2)^{\frac{1}{2}} (\sum_{i=1} |y_i|^2)^{\frac{1}{2}} \ni x_i, y_i \in \mathcal{R}$.

(11.4) **Theorem**: (**Minkokowsks Inequality**)

If $p \geq 1 \Longrightarrow (\sum_{i=1} |x_i + y_i|^p)^{1/p} \leq (\sum_{i=1} |x_i|^p)^{\frac{1}{p}} + (\sum_{i=1} |y_i|^p)^{\frac{1}{p}} \ni x_i, y_i \in \mathcal{R}$.

(11.5) **Example**: Let $X = C[0,1]$ represents a set of all bounded real functions and continuous on [0,1]. Define $d: X \times X \to \mathcal{R}$ by $d(f,g) = \int_0^1 |f(x) - g(x)| dx \; \forall f, g \in X$, then (X, d) be incomplete metric space.

(11.6) **Example**: Let $X = C[0,1]$ represents a set of all bounded real functions and continuous on [0,1]. Define $d: X \times X \to \mathcal{R}$ by $d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\} \forall f, g \in X$ then (X, d) be complete metric space.

(11.7) **Example**: Let $L^p[a,b], 1 \leq p \leq \infty$, represents a set of real functions $f: [a,b] \to \mathcal{R} \ni \int_0^1 |f(x)|^p dx < \infty$, this means $L^p[a,b] = \{ f: [a,b] \to \mathcal{R} \ni \int_0^1 |f(x)|^p dx < \infty \}$. Define $d: X \times X \to \mathcal{R}$ by $d(f,g) = (\int_0^1 |f(x) - g(x)|^p dx)^{1/p} \; \forall f, g \in L^p[a,b]$, then $(L^p[a,b], d)$ be complete metric.

(11.8) **Example**: Let $L^p, 1 \leq p \leq \infty$, represents a set of real sequences $x = \{x_n\} \ni \sum_{i=1} |x_i|^p < \infty$, this means $L^p = \{ x = \{x_n\} : \sum_{i=1} |x_i|^p < \infty\}$. Define $d: L^p \times L^p \to \mathcal{R}$ by $d_p(x,y) = (\sum_{i=1} |x_i - y_i|^p)^{1/p} \forall \; x = \{x_n\}, y = \{y_n\} \in L^p$, then (L^p, d_p) be complete metric.

Solution: the axioms 1,2,3 are clear.

(4) let $x, y \in L^p \Longrightarrow x + y = (x_1 + y_1, \ldots, x_n + y_n)$

$|x+y| = (\sum_{i=1} |x_i + y_i|^p)^{1/p}$

$|x+y| \leq (\sum_{i=1} |x_i|^p)^{\frac{1}{p}} + (\sum_{i=1} |y_i|^p)^{\frac{1}{p}} = |x| + |y|.$

(11.9) **Example**: Let L^∞,, represents a set of bounded real sequences $L^\infty = \{ x = \{x_n\} : n \in \mathbb{N}\}$. Define $d_\infty : L^\infty \times L^\infty \to \mathcal{R}$ by $d_\infty(x,y) =$ sup $\{\sum_{i=1} |x_i - y_i| : n \in \mathbb{N}$ $\forall\, x = \{x_n\}, y = \{y_n\} \in L^\infty$, then (L^∞, d_∞) be complete metric.