10. Convergence

(10.1) **Definition**: Let $\{x_n\}$ be a sequence in a metric space (X, d). We said that $\{x_n\}$ is a convergent in X, if $\exists x \in X \ni \forall \varepsilon > 0 \; \exists \; k \in \mathbb{Z}^+ \ni d(x_n, x) < \varepsilon \; \forall n > k$.

$\lim_{x \to \infty} x_n = x$ or $x_n \to x$ where $n \to \infty$

This means $d(x_n, x) \to 0 \Leftrightarrow x_n \to x$ where $n \to \infty$.

(10.2)**Theorem:** If $\{x_n\}$ is a convergent in (X, d), then the convergence point is a unique.

Proof: let $x_n \to x$ and $x_n \to y$ such that $x \neq y$.

Let $d(x, y) = \varepsilon \Longrightarrow \varepsilon > 0$

Since $x_n \to x \Longrightarrow \exists k_1 \in \mathbb{Z}^+ \ni d(x_n, x) < \frac{\varepsilon}{2} \; \forall n > k_1$

Since $x_n \to y \Longrightarrow \exists k_2 \in \mathbb{Z}^+ \ni d(x_n, y) < \frac{\varepsilon}{2} \; \forall n > k_2$

Put $k = \max \{k_1, k_2\} \Longrightarrow d(x_n, x) < \frac{\varepsilon}{2}, d(x_n, y) < \frac{\varepsilon}{2} \forall n > k$.

$\varepsilon = d(x, y) = d(x_n, x) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, but this is a contradiction $\Longrightarrow x = y$.

(10.3)**Example:** Let $\{x_n\}$ be a sequence in $(\mathcal{R}, d_u)$, since $\{x_n\}$ is a convergent

$\Longrightarrow \exists x \in \mathcal{R} \ni x_n \to x$

Let $\varepsilon > 0 \Longrightarrow \exists k \in \mathbb{Z}^+ \ni d(x_n, x) < \varepsilon \; \forall n > k$

Since $(\mathcal{R}, d_u)$ is an usual metric space $\Longrightarrow d(x_n, x) = |x_n - x|$

$|x_n - x| < \varepsilon \; \forall n > k \Longrightarrow -\varepsilon < x_n - x < \varepsilon \; \forall n > k \Longrightarrow x - \varepsilon < x_n < x + \varepsilon$

$\Longrightarrow x_n \in (x - \varepsilon, x + \varepsilon) \; \forall n > k$

This means $\{x_n\}$ is convergent in $\mathcal{R}$, if $\exists x \in \mathcal{R} \ni \forall \varepsilon > 0 \; \exists (x - \varepsilon, x + \varepsilon)$ with the center x.

(10.4)**Theorem:** Let A is a subset in (X, d), then $x \in \bar{A} \Leftrightarrow \exists \; \{x_n\}$ in $A \ni x_n \to x$.

(10.5) **Definition**: Let $\{x_n\}$ be a sequence in a metric space (X, d). We said that $\{x_n\}$ is Cauchy sequence in X, if $\forall \varepsilon > 0 \; \exists \; k \in \mathbb{Z}^+ \ni d(x_n, x_m) < \varepsilon \; \forall n, m > k$.

(10.6)**Theorem:** Every convergent sequence$\{x_n\}$ in a metric space (X, d) be Cauchy sequence.

Proof: let $\{x_n\}$ be a convergent sequence in $(X, d) \Longrightarrow \exists x \in X \ni x_n \to x$

Let $\varepsilon > 0$, since $x_n \to x \Longrightarrow \exists\, k \in \mathbb{Z}^+ \ni d(x_n, x) < \frac{\varepsilon}{2}\ \forall n > k$

If $n, m > k \Longrightarrow d(x_n, x) < \frac{\varepsilon}{2}, d(x_m, x) < \frac{\varepsilon}{2}$

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Longrightarrow \{x_n\}$ is Cauchy sequence.

(10.7)**Note:** Not necessary that every Cauchy sequence in a metric space (X, d) is a convergent, for example.

(10.8)**Example:** Let $X = \mathcal{R}\backslash\{0\}$, a function $d: X \times X \to \mathcal{R}$ defined by $d(x, y) = |x - y|$.

Solution: Let $x_n = \frac{1}{n}$, we note that (X, d) is a metric space and $\{x_n\}$ be Cauchy sequence in X, but $\{x_n\}$ does not convergent to x.

(10.9)**Theorem:** Let (X, d) a metric space and $\{x_n\}, \{y_n\}$ in $X \ni x_n \to x, y_n \to y \ni x, y \in X$, then $d(x_n, y_n) \to (x, y)$.

Proof: $d(x_n, y) - d(x, y) = (d(x_n, y_n) - d(x_n, y)) + (d(x_n, y) - d(x, y))$

$|d(x_n, y) - d(x, y)| \leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \to 0, n \to \infty \Longrightarrow d(x_n, y_n) \to (x, y)$.

(10.10)**Example:** Let (X, d) is a discrete metric space and $\{x_n\}$ in X. Prove that $x_n \to x \ni x \in X \Leftrightarrow \exists k \in \mathbb{Z}^+ \ni x_n = x\ \forall n > k$.

Proof: let $x_n \to x \Longrightarrow \forall \varepsilon > 0\ \exists\, k \in \mathbb{Z}^+ \ni d(x_n, x) < \varepsilon\ \forall n > k$.

Since $d(x, y) = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases} \Longrightarrow d(x_n, x) = 0\ \forall n > k \Longrightarrow x_n = x \forall n > k$.

(10.11)**Definition:** We said that (X, d) is complete, if for all Cauchy sequence is a convergent.

(10.12)**<u>Example:</u>** Euclidean space $(\mathcal{R}^n, d)$ be complete metric space.

Solution: let $x, y \in \mathcal{R}^n, \ni x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Let $\{x_m\}$ be Cauchy sequence in $\mathcal{R}^n$.

$x_m = \left(x_1{}^{(m)}, \dots, x_n{}^{(m)}\right), x_m \in \mathcal{R}^n$

Let $\varepsilon > 0 \implies \exists\ k \in \mathbb{Z}^+ \ni d(x_m, x_l) = \sqrt{\sum_{i=1}^{n} (x_i{}^{(m)} - y_i{}^{(l)})^2}$

$\implies \{x_m\}$ Cauchy sequence in $\mathcal{R}\ \forall i = 1, \dots, n$

Since $\mathcal{R}$ is complete field $\implies x_i \in \mathcal{R} \ni i = 1, \dots, n\ \forall\ x_i{}^{(m)} \to x_i$

Put $x = (x_1, \dots, x_n) \implies x \in \mathcal{R}^n \implies x_m \to x$

So, $\{x_m\}$ be convergent in $\mathcal{R}^n$.