## 9. Interior Points

(9.1) **Definition**: Let $(X, d)$ be a metric space, and $A \subseteq X$. We say that a point $x \in A$ is an interior point in $A$, if $\exists$ an open set $G$ in $X \ni x \in G \subset A$. Or if $\exists\, r > 0 \ni B_r(x) \subseteq A$.

(9.2) **Definition**: The set of all interior points in $A$ is denoted by

$A^\circ = \{x \in A: \exists r > 0, B_r(x) \subseteq A\}$, so $A^\circ \subset A$.

(9.3) **Notes**: From previous definitions, we deduce

1. $A^\circ$ is an open set in $X$.
2. $A$ is an open set in $X$ iff $A^\circ = A$.
3. $A^\circ = (A^\circ)^\circ$.

(9.4) **Example:** Let $(X, d)$ be an indiscrete metric space, and $A \subseteq X$. Calculate $A^\circ$.

**Solution:** since $(X, d)$ be an indiscrete metric space $\Longrightarrow d(x, y) = 0 \;\; \forall x, y \in X$.

$$\Longrightarrow A^\circ = \begin{cases} \emptyset, A \neq X \\ X, A = X \end{cases}$$

(9.5) **Example:** Let $(X, d)$ be a discrete metric space, and $A \subseteq X$. Calculate $A^\circ$.

**Solution:** since $(X, d)$ be a discrete metric space $\Longrightarrow A$ is an open set in $X$.

$\Longrightarrow A^\circ = A$.

(9.6) **Example:** Let $(\mathcal{R}, d_u)$ be usual metric space, and $A \subseteq \mathcal{R}$, then we have

1. If $A = (a, b) \Longrightarrow A^\circ = (a, b)$.
2. If $A = (a, b] \Longrightarrow A^\circ = (a, b)$.
3. If $A = [a, b) \Longrightarrow A^\circ = (a, b)$.
4. If $A = [a, b] \Longrightarrow A^\circ = (a, b)$.
5. If $A = [a, b] \cup [c, d] \Longrightarrow A^\circ = (a, b) \cup (c, d)$.
6. If $A$ is a finite, then $A^\circ = \emptyset$.
7. If $A = \mathbb{N} \Longrightarrow A^\circ = \emptyset$.
8. If $A = \mathbb{Z} \Longrightarrow A^\circ = \emptyset$.
9. If $A = \mathbb{Q} \Longrightarrow A^\circ = \emptyset$.
10. If $A = \{\frac{1}{n}, n \in \mathbb{N}\} \Longrightarrow A^\circ = \emptyset$.

(9.7) **Theorem:** Let $(X, d)$ be a metric space, and $A, B \subseteq X$, then we have

1. If $A \subseteq B \Longrightarrow A^\circ \subseteq B^\circ$.
2. $(A \cap B)^\circ = A^\circ \cap B^\circ$.
3. $A^\circ \cap B^\circ \subset (A \cup B)^\circ$.

**Proof:** (1) let $x \in A^\circ \Longrightarrow \exists r > 0 \ni B_r(x) \subseteq A$.

Since $A \subseteq B \Longrightarrow B_r(x) \subseteq B \Longrightarrow x \in B^\circ \Longrightarrow A^\circ \subseteq B^\circ$.

(9.8) **Note:** Not necessary that $(A \cup B)^\circ = A^\circ \cup B^\circ$, for example

Let $(\mathcal{R}, d_u)$ be usual metric space, and let $A = [0,1], B = [1,2]$, we have

$A^\circ = (0,1), B^\circ = (1,2) \Longrightarrow 1 \notin A^\circ, 1 \notin B^\circ \Longrightarrow 1 \notin A^\circ \cup B^\circ$, but $A \cup B = [0,2]$

$\Longrightarrow (A \cup B)^\circ = (0,2)$ and $1 \in (A \cup B)^\circ \Longrightarrow (A \cup B)^\circ \neq A^\circ \cup B^\circ$.

(9.9) **Definition:** Let $(X, d)$ be a metric space, and $A \subseteq X$. We said that $x \in X$ is a closure point of set $A$, if $\forall r > 0 \; \exists y \in A \ni d(x, y) < r$.

$\bar{A} = \{x \in X : \forall r > 0, \exists y \in A \ni d(x, y) < r\} \Longrightarrow A \subset \bar{A}$.

(9.10) **Notes:** From previous definitions, we deduce

1. $\bar{A}$ is a closed set in $X$.
2. $\bar{A}$ is a closed set $\Leftrightarrow \bar{A} = A$.
3. $\bar{A} = \bar{\bar{A}}$.

(9.11) **Example:** Let $(X, d)$ be indiscrete metric space, and $A \subseteq X$. Calculate $\bar{A}$.

**Solution:** since $(X, d)$ be indiscrete metric space $\Longrightarrow d(x, y) = 0 \; \forall x, y \in X$

$$\Longrightarrow \bar{A} = \begin{cases} \emptyset, A = \emptyset \\ X, A \neq \emptyset \end{cases}$$

(9.12) **Example:** Let $(X, d)$ be discrete metric space, and $A \subseteq X$. Calculate $\bar{A}$.

**Solution:** since $(X, d)$ be discrete metric space $\Longrightarrow A$ is a closed set in $X$.

$\Longrightarrow \bar{A} = A$.

(9.13) **Example:** Let $(\mathcal{R}, d_u)$ be usual metric space, and $A \subseteq \mathcal{R}$, then we have

1. If $A = (a, b) \Longrightarrow \bar{A} = [a, b]$.
2. If $A = (a, b] \Longrightarrow \bar{A} = [a, b]$.
3. If $A = [a, b) \Longrightarrow \bar{A} = [a, b]$.

4. If $A = [a,b] \Longrightarrow \bar{A} = [a,b]$.
5. If $A = \mathbb{N} \Longrightarrow \bar{A} = \mathbb{N}$.
6. If $A = \mathbb{Z} \Longrightarrow \bar{A} = \mathbb{Z}$.
7. If $A = \mathbb{Q} \Longrightarrow \bar{A} = \mathcal{R}$.
8. If $A = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \Longrightarrow \bar{A} = A \cup \{0\}$.
9. If $A = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\right\} \Longrightarrow \bar{A} = A \cup \{1\}$.

(9.14) **Theorem:** Let $(X,d)$ be a metric space, and $A, B \subseteq X$, then we have

1. If $A \subset B \Longrightarrow \bar{A} \subseteq \bar{B}$.
2. $\bar{A} \cap \bar{B} \subset \overline{(A \cap B)}$.
3. $\bar{A} \cap \bar{B} \subset \overline{(A \cup B)}$.

**Proof:** (1) let $x \in \bar{A} \Longrightarrow \forall r > 0 \exists y \in A \ni d(x,y) < r$.

Since $A \subset B \Longrightarrow y \in B \Longrightarrow \forall r > 0\ \exists y \in B \ni d(x,y) < r \Longrightarrow x \in \bar{B} \Longrightarrow \bar{A} \subseteq \bar{B}$.

(9.15) **Note:** Not necessary that $\bar{A} \cap \bar{B} = \overline{(A \cap B)}$, for example

Let $(\mathcal{R}, d_u)$ be usual metric space, and let $A = (0,1), B = (1,2)$, we have

$\bar{A} = [0,1], \bar{B} = [1,2] \Longrightarrow \bar{A} \cap \bar{B} = \{1\}$, but $A \cap B = \emptyset \Longrightarrow \overline{(A \cap B)} = \emptyset$

$\Longrightarrow \bar{A} \cap \bar{B} \neq \overline{(A \cap B)}$.

(9.16) **Definition:** Let $(X,d)$ be a metric space, and $A \subseteq X$. We said that $x \in X$ is a limit point of set $A$, if $\forall r > 0\ \exists y \in A \ni y \neq x, d(x,y) < r$. The set of all limit points denoted by $A' = \{x \in X : \forall r > 0, \exists y \in A \ni y \neq x, d(x,y) < r\}$.

(9.17) **Definition:** We said that $x \in X$ is an isolated point of $A$, if $x \in A, x \notin A'$.

(9.18) **Definition:** We said that the set $A$ is an isolated set, if $A \cap A' = \emptyset$.

(9.19) **Definition:** We said that the set $A$ is a perfect set, if $A = A'$.

(9.20) **Definition:** We said that the set $A$ is a dense set, if $A \subset A'$.

(9.21) **Theorem:** Let $(X,d)$ be a metric space, and $A \subseteq X$, then we have

1. If $A = \emptyset \Longrightarrow A' = \emptyset$.
2. If $x \in A' \Longrightarrow x \in (A \backslash \{x\})'$.
3. $A' \subset \bar{A}$.

4. $\bar{A} = A \cup A'$.
5. If $A' = \emptyset \Longrightarrow A$ is a closed in $X$.
6. $A$ is a closed in $X \Longleftrightarrow A' \subset A$.

**Proof:** (1) Since $\emptyset \cap (V\backslash\{x\}) = \emptyset\ \forall x \in X$ and $\forall$ neighborhood $V$ of $X \Longrightarrow \emptyset' = \emptyset$.

(9.22) **Example:** Let $(X, d)$ be indiscrete metric space, and $A \subseteq X$. Calculate $A'$.

**Solution:** (1) if $A = \emptyset \Longrightarrow A' = \emptyset$.

(2) if $A = X$, then we have

a. if $X$ contains one element $\Longrightarrow A' = \emptyset$.

b. if $X$ contains more than one element $\Longrightarrow A' = X$.

3. if $A \neq \emptyset, A \neq X$.

a. if $A$ contains one element $\Longrightarrow A = \{a\} \Longrightarrow A' = X\backslash\{a\}$.

b. if $A$ contains more than one element $\Longrightarrow A' = X$.

(9.23) **Theorem:** Let $(X, d)$ be a metric space, and $A, B \subseteq X$, then we have

1. If $A \subseteq B \Longrightarrow A' \subseteq B'$.
2. $(A \cap B)' \subset A' \cap B'$.
3. $(A \cup B)' = A' \cup B'$.

**Proof:** (1) let $x \in A' \Longrightarrow \forall r > 0\ \exists y \in A \ni y \neq x, d(x, y) < r$.

Since $A \subseteq B \Longrightarrow y \in B \Longrightarrow \forall r > 0\ \exists y \in B \ni y \neq x, d(x, y) < r$

$\Longrightarrow x \in B' \Longrightarrow A' \subseteq B'$.

(9.24) **Note:** Not necessary that $(A \cap B)' = A' \cap B'$, for example

Let $X = \{a, b, c, d\}, (X, d)$ be indiscrete metric space, if $A = \{a\}, B = \{c, d\}$.

$\Longrightarrow A' = \{b, c, d\}, B' = X \Longrightarrow A' \cap B' = \{b, c, d\} \Longrightarrow A \cap B = \emptyset \Longrightarrow (A \cap B)' = \emptyset$

$\Longrightarrow (A \cap B)' \neq A' \cap B'$.

(9.25) **Definition:** Let $(X, d)$ be a metric space, and $A \subseteq X$. We said that $x \in X$ is a boundary point of set $A$, if $\forall r > 0\ \exists z \in A^c, y \in A \ni\ d(x, z) < r, d(x, y) < r$. The set of all boundary points denoted by

$\partial(A) = \{x \in X: \forall r > 0, \exists y \in A, z \in A^c \ni, d(x,y) < r, d(x,z) < r\}$.

(9.26) **Theorem:** Let $(X,d)$ be a metric space, and $A \subseteq X$, then

1. $\partial(A) = \bar{A} \cap (\overline{A^c}) \Longrightarrow \partial(A) \subset \bar{A}$.
2. $\partial(A) = \partial(A^c)$.
3. $\partial(A)$ is a closed set in $X$.
4. $A^\circ = A \backslash \partial(A)$.
5. $\bar{A} = A \cup \partial(A)$.

**Proof:** (1) let $x \in \partial(A) \Longrightarrow \forall$ open set $G$ in $X$ and $x \in G$

$\Longrightarrow G \cap A \neq \emptyset, G \cap A^c \neq \emptyset \Longrightarrow x \in \bar{A}, x \in \overline{A^c} \Longrightarrow x \in \bar{A} \cap (\overline{A^c})$

$\Longrightarrow \partial(A) \subseteq \bar{A} \cap (\overline{A^c})$

By same way we prove that $\bar{A} \cap (\overline{A^c}) \subseteq \partial(A) \Longrightarrow \partial(A) = \bar{A} \cap (\overline{A^c})$.

(9.27) **Example:** Let $(X,d)$ be discrete metric space, and $A \subseteq X$. Calculate $\partial(A)$.

**Solution:** since $A$ is a closed $\Longrightarrow A = \bar{A}$ and $A^c$ is closed $\Longrightarrow \overline{A^c} = A^c$

$\Longrightarrow \partial(A) = \bar{A} \cap (\overline{A^c}) = A \cap A^c = \emptyset$.