7. Pseudo- Metric Function

(7.1) **Definition**: If X be a non-empty set. We said that a function $d: X \times X \to \mathcal{R}$ be pseudo- metric function, if

1. $d(x,y) \geq 0 \ \forall x,y \in X$.
2. $d(x,x) = 0 \ \forall x \in X$.
3. $d(x,y) = d(y,x) \ \forall x,y \in X$.
4. $d(x,y) \leq d(x,z) + d(z,y) \ \forall x,y,z \in X$.

(7.2) **Definition**: A function d on X be a metric, if $d(x,y) \neq 0 \ \forall x \neq y$.

(7.3) **Definition**: Pseudo- metric space is (X,d), such that X is a non-empty set and d is Pseudo- metric function on X.

(7.4) **Theorem:** Let (X,d) be Pseudo- metric function. We define a relation $\sim$ on X as $x \sim y \Leftrightarrow d(x,y) = 0$, then

1. $\sim$ be an equivalent relation on X.
2. If $[x]$ be an equivalent class of $x$and $A = \{[x]: x \in X\}$, then $d^*: A \times A \to \mathcal{R}$ defined by $d^*([x],[y]) = d(x,y)$ is a metric on A, this means (A, d^*) be a metric space.

Proof: (1) since $d(x,x) = 0 \ \forall x \in X \Rightarrow x \sim x \Rightarrow \sim$ is a reflexive.

Let $x \sim y \Rightarrow d(x,y) = 0$, but $d(x,y) = d(y,x) \Rightarrow d(y,x) = 0 \Rightarrow y \sim x \Rightarrow \sim$ is a symmetric.

Let $y \sim z, x \sim y \Rightarrow d(x,y) = 0,\ d(y,z) = 0,$ since $d(x,z) \leq d(x,y) + d(y,z) \Rightarrow d(x,z) \leq 0$, but $d(x,z) \geq 0 \Rightarrow d(x,z) = 0 \Rightarrow x \sim z \Rightarrow \sim$ is a transitive

$\Rightarrow \sim$ is an equivalent relation on X.

(7.5)**Example:** Let X be a set of all real functions on [0,1], we define $d: X \times X \to \mathcal{R}$ by $d(f,g) = \int_0^1 |f(x) - g(x)| dx \ \forall f,g \in X$, then d be Pseudo- metric and does not metric on X.

Solution: (1) let $f,g \in X \Rightarrow |f(x) - g(x)| > 0$

$\Rightarrow d(f,g) = \int_0^1 |f(x) - g(x)| dx > 0.$

(2) let $f \in X$, $d(f,f) = \int_0^1 |f(x) - f(x)| dx = \int_0^1 |0|\, dx = 0.$

(3) let $f, g \in X$

$d(f,g) = \int_0^1 |f(x) - g(x)|dx = \int_0^1 |g(x) - f(x)|dx = d(g,f).$

(4) let $f, g, h \in X$

$$d(f,g) = \int_0^1 |f(x) - g(x)|dx$$

$$= \int_0^1 |f(x) - h(x) + h(x) - g(x)|dx \leq \int_0^1 |f(x) - h(x)| + |h(x) - g(x)|dx$$

$$= \int_0^1 |f(x) - h(x)|dx + \int_0^1 |h(x) - g(x)|dx = d(f,h) + d(h,g)$$

$\Longrightarrow d$ Pseudo- metric.

Let $f, g: [0,1] \to \mathcal{R}$ defined by

$f(x) = \begin{cases} 2 & , x = 0 \\ 0, & 0 < x \leq 1 \end{cases}$ and $g(x) = \begin{cases} 1 & , x = 0 \\ 0, & 0 < x \leq 1 \end{cases}$

$f(x) - g(x) = \begin{cases} 1 & , x = 0 \\ 0 & , 0 < x \leq 1 \end{cases} \Longrightarrow d(f,g) = \int_0^1 |f(x) - g(x)|dx = 0$, but $f \neq g$.

Product Space

(7.6) **Definition:** Let X, Y be Cartesian product of X, Y denoted by $X \times Y$and defined by $X \times Y = \{(x,y): x \in X, y \in Y\}$, its clear $X \times Y \neq Y \times X$, if $X \times Y \neq \emptyset$, then $Y \times X \neq \emptyset$.

(7.7) **Example:** If $(X, d_1), (Y, d_2)$ be a metric spaces, then $(X \times Y, d)$ be a metric space, such that $d\big((x_1, y_1), (x_2, y_2)\big) =$ max $\{d_1(x_1, x_2), d_2(y_1, y_2)\}\ \forall (x_1, y_1), (x_2, y_2) \in X \times Y$.

Solution: (1) let $(x_1, y_1), (x_2, y_2) \in X \times Y \Longrightarrow d_1(x_1, x_2) \geq 0, d_2(y_1, y_2) \geq 0 \Longrightarrow$ max $\{d_1(x_1, x_2), d_2(y_1, y_2)\} \geq 0 \Longrightarrow d\big((x_1, y_1), (x_2, y_2)\big) \geq 0$.

(2) let $(x_1, y_1), (x_2, y_2) \in X \times Y$

$d\big((x_1,y_1),(x_2,y_2)\big)=0 \Leftrightarrow \max\{d_1(x_1,x_2),d_2(y_1,y_2)\}=0 \Leftrightarrow d_1(x_1,x_2)=0, d_2(y_1,y_2)=0 \Leftrightarrow (x_1,y_1)=(x_2,y_2) \Leftrightarrow x_1=x_2,\ \ y_1=y_2.$

(3) let $(x_1,y_1),(x_2,y_2)\in X\times Y$

$d\big((x_1,y_1),(x_2,y_2)\big)=\max\{d_1(x_1,x_2),d_2(y_1,y_2)\}=\max\{d_2(x_1,x_2),d_1(y_1,y_2)\}=d\big((x_2,y_2),(x_1,y_1)\big).$

(4) let $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X\times Y$

$d\big((x_1,y_1),(x_2,y_2)\big)=\max\{d_1(x_1,x_2),d_2(y_1,y_2)\}\le\max\{d_1(x_1,x_3)+d_1(x_3,x_2),d_2(y_1,y_3)+d_2(y_3,y_2)\}\le\max\{d_1(x_1,x_3),d_2(y_1,y_3)\}+\max\{d_1(x_3,x_2),d_2(y_3,y_2)\}=d\big((x_1,y_1),(x_3,y_3)\big)+d\big((x_3,y_3),(x_2,y_2)\big).$

Euclidean Spaces

(7.8) **<u>Definition</u>:** Let $n\in\mathbb{Z}^+$, a finite sequence $(x_1,\dots,x_n)$ consists of n real numbers called n-tuples. We said a set which its elements n of components is Euclidean n-tuples and denoted by $\mathcal{R}^n \Rightarrow \mathcal{R}^n=\{(x_1,\dots,x_n):x_i\in\mathcal{R}, i=1,\dots,n\}$

$(x_1,\dots,x_n)+(y_1,\dots,y_n)=(x_1+y_1,\dots,x_n+y_n)$

$\lambda(x_1,\dots,x_n)=(\lambda x_1,\dots,\lambda x_n)\ \forall\lambda\in\mathcal{R},\forall x=(x_1,\dots,x_n),y=(y_1,\dots,y_n)\in\mathcal{R}^n.$

(7.9) **<u>Example:</u>** Let a function $d:\mathcal{R}^n\times\mathcal{R}^n\to\mathcal{R}$ defined by

$d(x,y)=(\sum_{i=1}^n(x_i-y_i)^2)^{1/2}\forall x=(x_1,\dots,x_n),y=(y_1,\dots,y_n)\in\mathcal{R}^n$, then d be a metric function on $\mathcal{R}^n$.

Solution: (1) let $x=(x_1,\dots,x_n),y=(y_1,\dots,y_n)\in\mathcal{R}^n$

$\Rightarrow x_i-y_i\in\mathcal{R}\ \ \forall i=1,2,\dots,n \Rightarrow (x_i-y_i)^2\ge 0\ \ \forall i=1,2,\dots,n \Rightarrow d(x,y)\ge 0.$

(2) $d(x,y)=0 \Leftrightarrow (\sum_{i=1}^n(x_i-y_i)^2)^{\frac{1}{2}}=0 \Leftrightarrow \sum_{i=1}^n(x_i-y_i)^2=0 \Leftrightarrow (x_i-y_i)^2=0\ \ \forall i=1,\dots,n \Leftrightarrow x_i-y_i=0\ \forall i=1,\dots,n \Leftrightarrow x_i=y_i\ \ \forall i=1,\dots,n \Leftrightarrow x=y.$

(3) let $x=(x_1,\dots,x_n),y=(y_1,\dots,y_n)\in\mathcal{R}^n$

$d(x,y)=(\sum_{i=1}^n(x_i-y_i)^2)^{\frac{1}{2}}=(\sum_{i=1}^n(y_i-x_i)^2)^{\frac{1}{2}}=d(y,x).$

(4) $x=(x_1,\dots,x_n),y=(y_1,\dots,y_n),z=(z_1,\dots,z_n)\in\mathcal{R}^n$

Put $\alpha_i=x_i-z_i$ and $\beta_i=z_i-y_i$

$d(x,z) = (\sum_{i=1}^{n} (x_i - z_i)^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} {\alpha_i}^2)^{\frac{1}{2}}$

$d(z,y) = (\sum_{i=1}^{n} (z_i - y_i)^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} {\beta_i}^2)^{\frac{1}{2}}$

$$d(x,y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} ((x_i - z_i) + (z_i - y_i))^2)^{\frac{1}{2}}$$
$$= (\sum_{i=1}^{n} (\alpha_i + \beta_i)^2)^{\frac{1}{2}}$$

$(\sum_{i=1}^{n} (\alpha_i + \beta_i)^2)^{\frac{1}{2}} \leq (\sum_{i=1}^{n} {\alpha_i}^2)^{\frac{1}{2}} + (\sum_{i=1}^{n} {\beta_i}^2)^{\frac{1}{2}}$

$\Longrightarrow d(x,y) \leq d(x,z) + d(z,y) \Longrightarrow d$ is a metric function on $\mathcal{R}^n$.

(7.10) **Example:** Let a function $d: \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}$ defined by

$d(x,y) = \sum_{i=1}^{n} |x_i - y_i| \; \forall x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathcal{R}^n$, then d be a metric function on $\mathcal{R}^n$.

Solution: (1), (2), (3) are clear.

(4) let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n), z = (z_1, \dots, z_n) \in \mathcal{R}^n$

Put $\alpha_i = x_i - z_i$ and $\beta_i = z_i - y_i$

$d(x,z) = \sum_{i=1}^{n} |\alpha_i|, d(z,y) = \sum_{i=1}^{n} |\beta_i|$

$\Longrightarrow d(x,y) = \sum_{i=1}^{n} |\alpha_i - \beta_i|$

Since $|\alpha_i - \beta_i| \leq |\alpha_i| + |\beta_i|$

$\Longrightarrow d(x,y) \leq d(x,z) + d(y,z)$

$\Longrightarrow d$ is a metric function on $\mathcal{R}^n$.