4. Infinite Series

Let $\{a_n\}$ a real sequence and $S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3, \dots, S_n = a_1 + a_2 + \cdots + a_n$. A sequence of partial sums $\{S_n\}$ is called an infinite series and its denoted by $\sum_{n=1}^{\infty} a_n$. We say that $a_1, a_2, a_3, \dots$ be a terms of infinite series $\sum_{n=1}^{\infty} a_n$ and called of numbers $S_1, S_2, S_3, \dots$ be a partial sums of infinite series $\sum_{n=1}^{\infty} a_n$.

(4.1) **Definition**: Let $\{a_n\}$ be a real sequence and $S_n = \sum_{k=1}^{n} a_k$, we called of $\{S_n\}$ is an infinite series and denoted by $\sum_{n=1}^{\infty} a_n$.

(4.2) **Definition**: We say that $\sum_{n=1}^{\infty} a_n$ is a convergent, if $\{S_n\}$ is a converge to S, this means $(\lim_{n\to\infty} S_n = S)$, S is called infinite series sum $\sum_{n=1}^{\infty} a_n$, this means $S = \sum_{n=1}^{\infty} a_n$. If $\{S_n\}$ is a divergent (i.e. $\lim_{n\to\infty} S_n$ does not exist).

(4.3) **Example**: Does $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ convergent ?

$a_n = \frac{1}{n(n+1)}, S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} (\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{n+1} \Rightarrow S_n \to 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ and then $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is a convergent.

(4.4) **Theorem**: (some special infinite series)

1. $\sum_{n=1}^{\infty} ar^{n-1} \ni a \neq 0, r \neq 0$ is called geometric series and r is a basis of series. $\sum_{n=1}^{\infty} ar^{n-1}$ is a convergent, if $|r| < 1, S = \frac{a}{1-r}$ and otherwise its be a divergent.
2. $\sum_{n=1}^{\infty} \frac{1}{n}$ is called a harmonic series and it's a divergent.

Proof: (1) if $r = 1 \Rightarrow S_n = a + a + \cdots + a = na \Rightarrow \{na\}$ does not convergent, if it's a convergent, so it's a bounded, this means $\exists\, M \in \mathcal{R}^+ \ni |na| \leq M\ \forall n \in \mathbb{Z}^+ \Rightarrow n|a| \leq M \Rightarrow n \leq \frac{M}{|a|}\ \forall n \in \mathbb{Z}^+$, but this a contradiction (Archimedes property) $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ is a divergent.

(2) $a_n = \frac{1}{n}, S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, S_{2n} = \sum_{k=1}^{2n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} \Rightarrow S_{2n} - S_n \geq \frac{1}{2}\ \forall n \in \mathbb{Z}^+$, i.e. if $m = 2n, n \geq 1 \Rightarrow |S_m - S_n| \geq \frac{1}{2} \forall n, m \in \mathbb{Z}^+ \Rightarrow \{S_n\}$ does not Cauchy sequence $\Rightarrow \{S_n\}$ does not convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent.

(4.5) **Examples**:

1. $\sum_{n=0}^{\infty}\frac{1}{2^n}$ is a convergent, since $r=\frac{1}{2}$ and $\sum_{n=1}^{\infty}\frac{1}{2^n}=2$.
2. $\sum_{n=1}^{\infty}4^{n-1}$ is a divergent, since $r=4$.
3. $\sum_{n=1}^{\infty}(-\frac{1}{6})^{n-1}$ is a convergent, since $r=-\frac{1}{6}$ and $\sum_{n=1}^{\infty}(-\frac{1}{6})^{n-1}=\frac{6}{7}$.
4. $0.1+0.01+0.001+\cdots$ is a convergent, since $0.1+0.01+0.001+\cdots=\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots=\sum_{n=1}^{\infty}\frac{1}{10^n}=\sum_{n=1}^{\infty}\frac{1}{10}.\frac{1}{10^{n-1}}\Rightarrow a=\frac{1}{10},r=\frac{1}{10}\Rightarrow \sum_{n=1}^{\infty}\frac{1}{10^n}=\frac{1}{9}$.
5. The number $0.16666\ldots$ is a convergent, let $0.1\overline{6}=0.16666\ldots=0.1+0.06+0.006+0.0006+\cdots=0.1+\sum_{n=1}^{\infty}\frac{6}{10^{n+1}}=\sum_{n=1}^{\infty}\frac{6}{100}.\frac{1}{10^{n-1}}\Rightarrow a=\frac{6}{100},r=\frac{1}{10}\Rightarrow 0.1\overline{6}=0.1+\sum_{n=1}^{\infty}\frac{1}{10^{n+1}}=\frac{1}{15}$.

(4.6) **Theorem**: Let $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be a convergent infinite series, then

1. $\sum_{n=1}^{\infty}(a_n+b_n)$ is a convergent and $\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$.
2. $\sum_{n=1}^{\infty}\lambda a_n$ is a convergent $\forall\lambda\in\mathcal{R}$ and $\sum_{n=1}^{\infty}\lambda a_n=\lambda\sum_{n=1}^{\infty}a_n$.

Proof: (1) Let $S_n=\sum_{k=1}^{\infty}a_k$ and $T_n=\sum_{k=1}^{\infty}b_k$, since $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be a convergent infinite series $\Rightarrow$ $\sum_{n=1}^{\infty}a_n=S,\sum_{n=1}^{\infty}b_n=T\Rightarrow\{S_n\},\{T_n\}$ be a convergent $\Rightarrow S_n\to S,T_n\to T$ $\Rightarrow S_n+T_n\to S+T$, $S_n+T_n=\sum_{n=1}^{\infty}(a_n+b_n)\to S+T\Rightarrow\sum_{n=1}^{\infty}(a_n+b_n)=S+T=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$.

(4.7) **Corollary**: If $\sum_{n=1}^{\infty}a_n$ is a convergent and $\sum_{n=1}^{\infty}b_n$ is a divergent, then

1. $\sum_{n=1}^{\infty}(a_n+b_n)$ is a divergent.
2. $\sum_{n=1}^{\infty}\lambda b_n$ is a divergent $\forall\lambda\neq 0$.

Proof: (1) Suppose that $\sum_{n=1}^{\infty}(a_n+b_n)$ is a convergent, since $\sum_{n=1}^{\infty}a_n$ is a convergent $\Rightarrow$ $-\sum_{n=1}^{\infty}a_n$ is a convergent.

Since $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}(a_n+b_n-a_n)$ is a convergent, but this is a contradiction.

(4.8) **Example**: $\sum_{n=1}^{\infty}\frac{1}{n}$ and $-\sum_{n=1}^{\infty}\frac{1}{n}$ are a divergent, but $\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n})=\sum_{n=1}^{\infty}0$ is a convergent.